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TECHNICAL REPORT

EFFECTS OF THE RESPONSE PROCESS IN SEARCH MODELS WITH FALSE DETECTIONS

RONALD LOYE KRONZ

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Technical Report

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FOREWARD

Under Contract No. N00014-67-0181-0012 with the Office of Naval Research, the Systems Research Laboratory (SRL) has been conducting a research program to develop analytic models of defense processes, principally the combat process. A detailed description of all the research performed on this program through June 1970 was reported in SRL 2147 TR 70-2(U) "Development of Models for Defense Systems Planning" dated September 1970. Additional work related to the combat allocation process was reported in SRL 2147 TR 71-1 (U) "Development of Optimal Strategies in Heterogeneous Lanchester-Type Processes" dated June 1971.

The work in descriptive modeling of combat processes and the development of optimal weapon allocation strategies assumed perfect intelligence gathering capabilities of the forces. For this reason some of the research effort has been directed to the study of intelligence and reconnaissance processes. A literature review of this area (reported in SRL 2147 TR 70-1 "A Review of Search and Reconnaissance Theory Literature" dated January 1970) indicated the need to consider more realistically both environmental effects and search objectives (interaction with the combat process) in developing descriptive structures of the search process and analysis of optimal search

strategies. Some initial ideas in these directions were presented in SRL 2147 TR 70-2 (U). Research to examine one dimension of the environmental effects -- the visibility process -- has been performed and is being documented in the forthcoming report SRL 2147 TR 71-3 (U) "A Characterization of the Visibility Process and Its Effect on Search Policies." Research on incorporating the effects of search objectives (called the response process) is described in ~~this~~ the report. The research explores the development of mathematical structures which link the search and response processes, examines the effect that the response process has on classical search strategies, and attempts to develop some physical insight into the relationship between the two processes.

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CHAPTER 1

INTRODUCTION

Classical search theory consists of the collection of mathematical models which are used to determine preferred strategies for locating or finding an object of interest. These models may be viewed as experimentation strategy models where the experimentation process is modeled by a stochastic detection function with the underlying state of the system unaltered by the experimentation strategy. The objective of the activity is to find the object or target of interest.[1]

Associated with important practical search operations are experimentation costs. Thus, it is reasonable to assume that the searcher expects to benefit by using the information generated by the search. For the searcher to benefit from the search, he must be involved in some activity which is dependent on the search outcome. Military units search for targets to track their movement, engage them in battle, or perform some other activity which depends on first acquiring the target. The searcher in oil explorations attempts to locate oil deposits which would be profitable to recover and sell. In general, the searcher searches for the object of inter-

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[1]This is in contrast to stochastic control theory where the state of the process is altered by the controls and the objective is to estimate the state of the system.

est in order to optimize his involvement in this search related activity which we shall refer to as response process.

The purpose of this research is to explore the development of mathematical structures which link the search and response processes , examine the effect that the response process has on classical search strategies, and gain some physical insight into the relationship between the two processes. The specific research problems addressed are delineated in Section 1.3 following a general characterization of search models in Section 1.1 and a review of the search theory literature in Section 1.2.

1.1 *Characterization of Search Models*

Search problems may be structured with the elements and interactions indicated in Figure 1.1. The search-response system consists of detectors, responders and the decision maker. The search-response environment consists of targets (objects of the search), other elements and the response process. Interactions between the search-response system and its environment are separated into two main groupings called the search process and the response process. The search process consists of the information gathering interactions. The primary

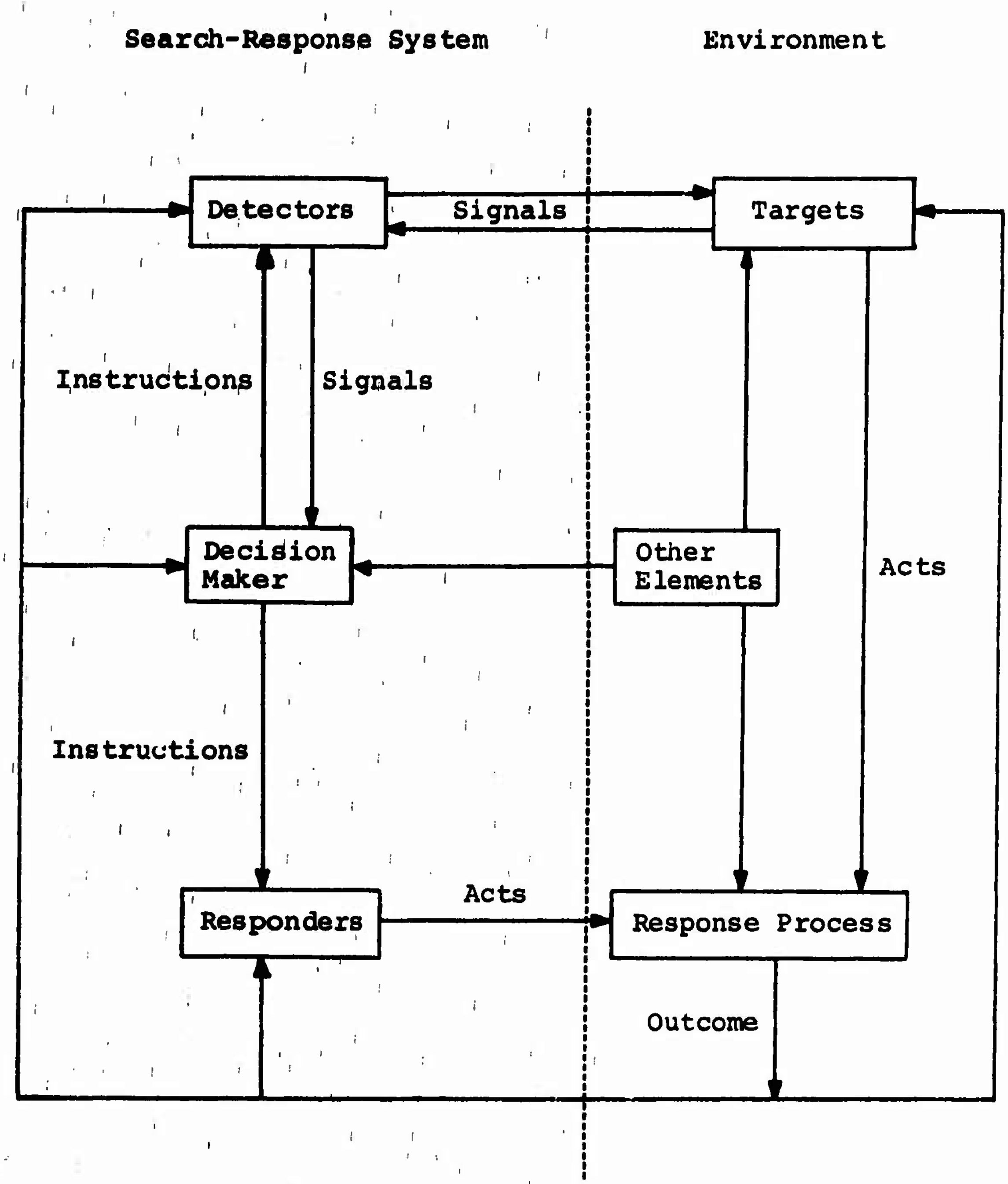

Figure 1.1 Search-Response System Diagram

search-related interactions of interest to the decision maker comprise the response process.

A target may be defined as an entity about which the search-response system attempts to gather information. A detector may be defined as a specialized device for collecting information about targets. Detectors receive signals which may contain desired target information, amplify these signals and transmit them to the decision maker. In some systems, such as radar, the detectors also provide the physical energy for the signals.

Those system elements which participate in controlling the operation of a search-response system are collectively called the decision maker. The decision maker extracts information from the detector signals and other sources and integrates this information to arrive at operating decisions for the action elements of the search-response system. The action elements which are not detectors are called responders. That is, an element which carries out a non-detection function under the control of the decision maker is defined to be a responder. While detectors may be involved in the response process, responders are the specialized elements having direct response process participation as their primary function.

The definitions given for target, detector, decision maker and responder are all operational in character. There-

fore, there may or may not be a simple correspondence between these operational elements and specific items of equipment for a system. Some of the elements in Figure 1.1 may be combined in one item. For example, a combat soldier may conduct a visual search for enemy soldiers, act as decision maker in evaluating the search information, and then perform the role of responder by engaging the target (enemy soldier) in a duel.

In Figure 1.1 the nature of the interactions as well as the elements themselves may vary for different problems. For example, radar detection systems transmit series of electromagnetic pulses into the environment and then collect reflected signals containing environmental information. But, unaided human visual detection systems simply collect a continuous stream of general scattered light containing environmental information.

A wide variety of problems including those of military combat, medical diagnosis, information storage and retrieval systems design, mineral exploration, and many others involve the operation of detection systems to collect information to aid in making basis system operating decisions. Thus, these problems may be considered to be search-response problems. Figure 1.2 presents a taxonomy for classifying search-response processes based on distinctions which we conjecture to be important in the analysis of such processes. Each possible path from block A to block C in Figure 1.2

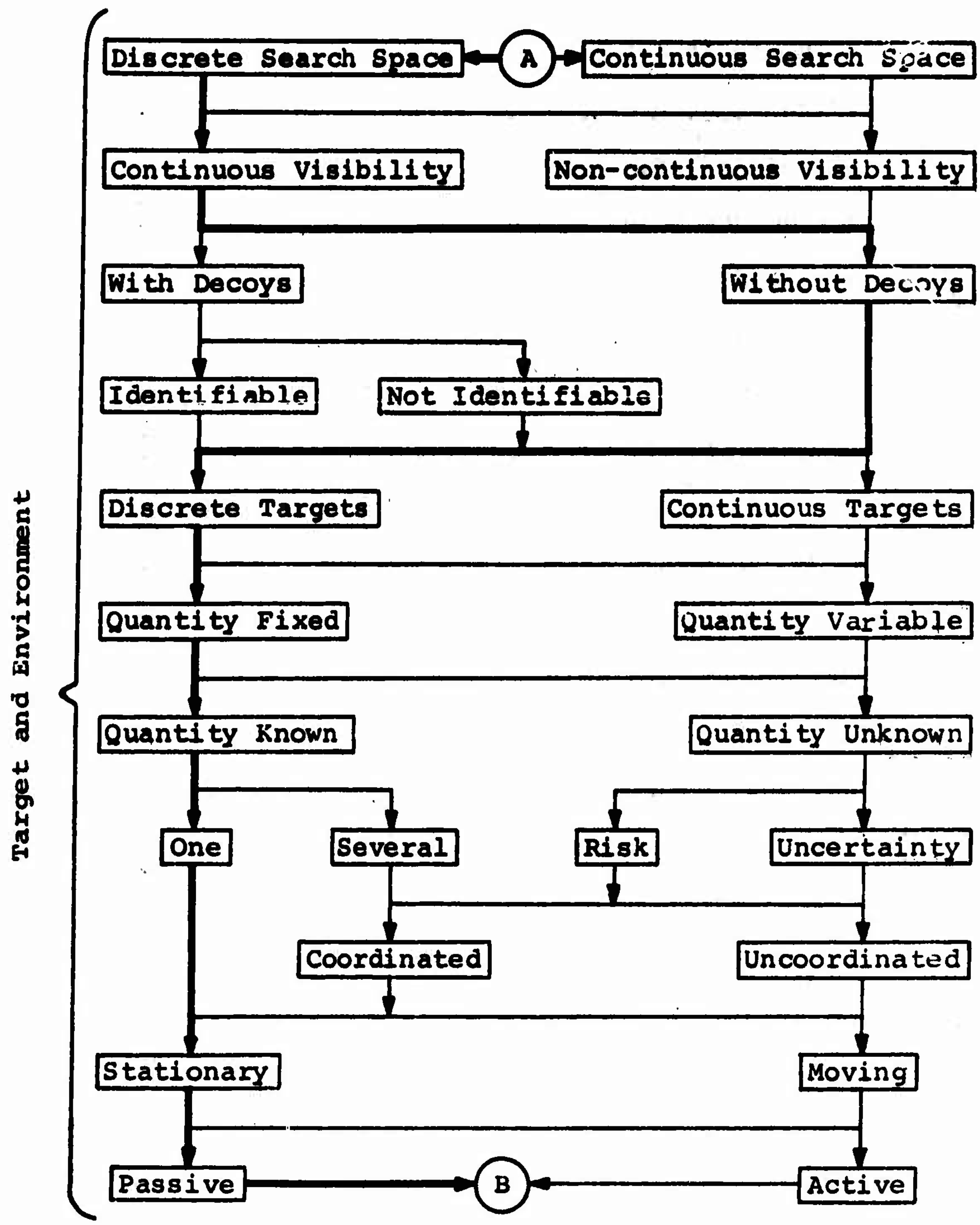

Figure 1.2 Search-Response Process Characterization

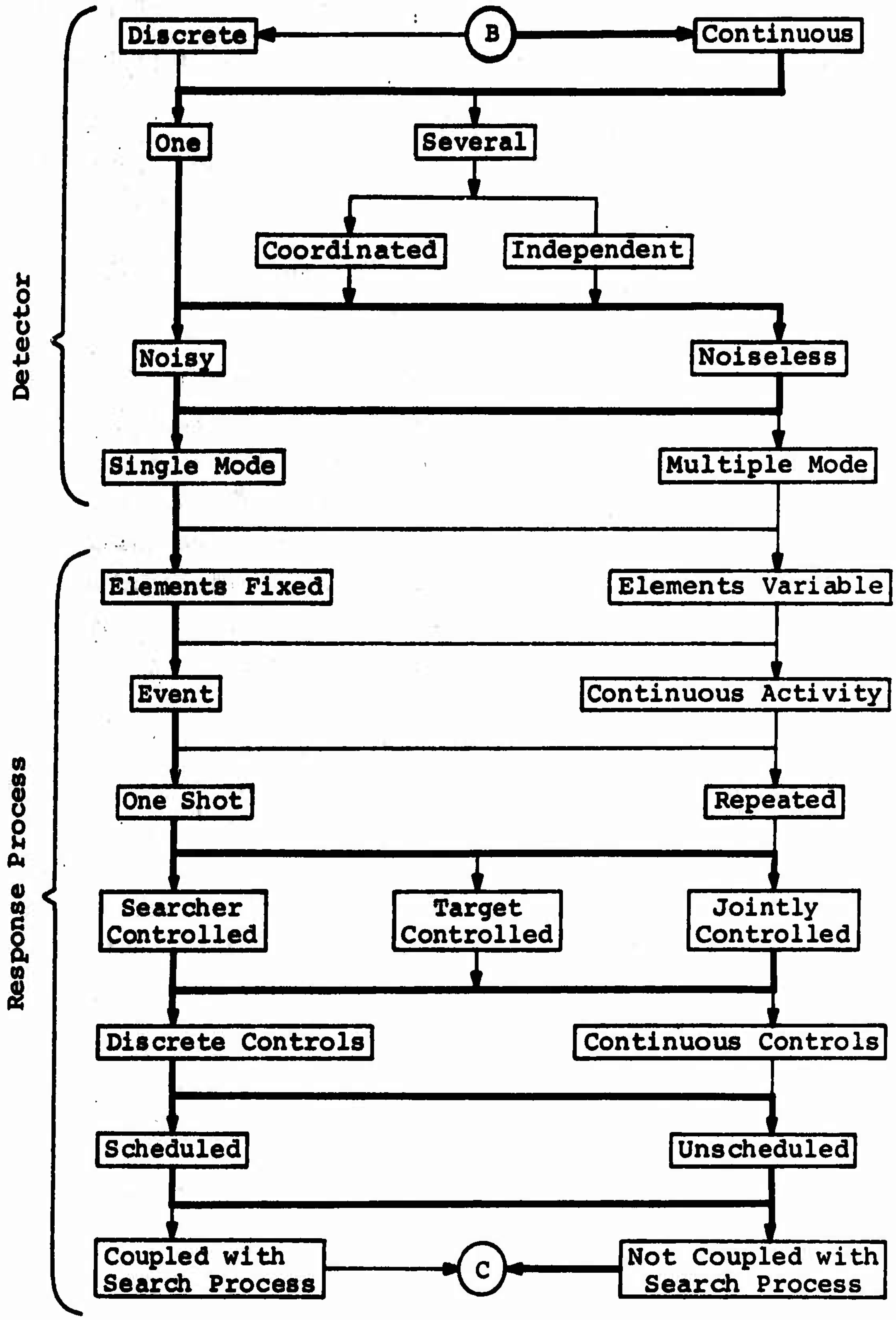

Figure 1.2 Search-Response Process Characterization

represents a class of search-response system process. Within each class there are many types of problems depending on the detailed characteristics of the process elements and relationships. The heavy-line paths through Figure 1.2 characterize the processes considered in this research.

The following comments are intended to amplify the meaning of certain elements in Figure 1.2:

1. In some search models the relevant target status is described by an integer representing the identification of the target status with one of a countable set of classifications. Such models are said to be discrete search space models. When the target status cannot be described by an integer, the search space is said to be continuous. For example, many medical diagnostic tests, which can be viewed as search processes, are designed to determine if the patient has a particular disease condition. The corresponding "search space" consists of the set of possible disease conditions which can be considered countable. In contrast, the combat soldier searching for enemy soldiers needs to know the enemy soldier location in order to fire his weapon (respond) optimally. Thus, his search space is continuous.

2. In searching for a target in real space the detector may be totally ineffective for some periods of time because of environmental obstructions such as trees, hills, etc. When this is the case, the target is said to be invisible. When no such effects interfere with the detection process, the target is said to be continuously visible.

3. Decoys are environmental items which produce detector signals similar to those produced by tar-

gets. For example, civilians at the scene of an infantry "search and destroy " operation are decoys. Identifiable decoys are those which can be identified as non-target items by the application of additional search effort.

4. In searching for discrete targets one assumes that a single target is detectable and significant. For continuous targets, however, the basic unit is sufficiently small to be either undetectable or unimportant. Only some larger quantity, representable by a continuous variable, is sought. An example of a continuous target is found in oil exploration searching.

5. For an unknown target quantity model the term risk refers to a situation in which an acceptable probabilistic description for the target quantity is assumed. The term uncertainty applies to those situations for which no acceptable probabilistic description for the target quantity is known. This distinction, though only a matter of what the decision maker is willing to assume for analytic purposes, is quite important in determining the form of the analysis and usefulness of the results.

6. Active targets select their behavior (location, movement, etc.) for the purpose of gaining advantage in their involvement in the search-response process. The behavior of passive targets lacks such purpose.

7. The probability distribution of effort (or time) until a detector detects a target which is present is called the detection function. If this proba-

bility distribution is discrete, the detector is said to be a discrete detector. If the probability distribution is continuous, the detector is said to be continuous.

8. Noisy detectors are those which transmit signals containing false positive indications of target information. That is, noisy detectors sometimes trigger false "detections." In contrast, noiseless detectors never yield unreliable or false indications of target information.

9. Some detection systems alter their mode of operation when a positive indication of a target is encountered. Such detection systems may be called multiple mode systems. Systems which do not switch from one mode to another may be called single mode detection systems.

10. Response processes can be classified according to whether the time duration of the process is important to the problem. Events are processes which, for practical purposes, consume no time. Continuous activities are processes which consume a significant amount of time.

11. The control of a response process can involve either the responder, the target or both, as well as other environmental elements. In the case of a duel between opposing infantry combat patrols either unit may be able to disengage from the duel. Thus, both the search-responder and the target participate in controlling the response (duel) process.

12. For some search problems the response process occurs regardless of the search outcome. In

such cases the response process can be thought of as scheduled. For example, in an air force tactical support operation related to a ground battle, bombers may be used regularly to attack the most valuable targets identified from photographs made by reconnaissance aircraft. The detectors are the reconnaissance aircraft; the responders are the bombers. The purpose of the search is to improve the effectiveness of the bombing.

For some other problems the response process cannot operate unless a detection occurs. In this event the response process may be thought of as unscheduled. The duel between opposing infantry patrols represents an unscheduled response process.

13. Coupling between the search and response process exists if either of these processes directly interrupts, interferes, degrades or enhances the other.

1.2 Review of Search Theory Literature

In this section we review some of the important search theory developments which are relevant to the search models developed in this research. Emphasis is placed on those works which are germane to the research. For a more comprehensive review of the search theory literature see Dobbie (1968) and Moore (1970).

Analyzing World War II antisubmarine operations Koopman (1946) developed a descriptive model for computing the detection function for a continuous search

space, continuous detector search situation. He assumed that the underlying detection process is a Poisson process in time with intensity parameter, $\gamma(t)$, a function only of the distance from the detector to the target. Thus, the probability that the detector will fail to detect the target during the interval (t_0, t_1) is

$$\bar{P}(t_0, t_1) = \exp\left[-\int_{t_0}^{t_1} \gamma(t)dt\right] \tag{1}$$

where $\gamma(t)$ is evaluated along the trajectory describing the position of the target relative to the detector as a function of time.

Equation 1 cannot be applied directly to any search problem because the relative trajectory of the target is never known to a searcher. However, by making various simplifying assumptions, useful detection models can be derived. If the relative trajectory is a particular straight line, (1) can be used to compute the probability of detecting the target as a function of the minimum distance between the detector and target (miss distance) and the relative speed. (This function, with relative speed fixed, is called the lateral range curve.) Given any probability distribution of miss distances, one can

obtain the probability distributions of miss distances, ranges and bearings for the targets that are detected.

Koopman considered the following search problem:

1. A fixed target's position has a uniform probability distribution over a large area of size A.
2. The detector is a definite range law device. That is, if r is the distance between the detector and the target,

$$\gamma = \begin{cases} 0 & \text{for } 0 \leq r < R_m \\ \infty & \text{for } R_m \leq r, \end{cases}$$

   where R_m is the range of the detector.
3. The detector moves with constant speed along a random path in A consisting of straight line segments which are much longer than R_m. Then, it can be shown that the detection process follows the "law of random search"

$$\bar{P}(L) = \exp\left[-\frac{WL}{A}\right], \tag{2}$$

   where

   L = length of the track covered by the detector, and

   W = the integral of the lateral range curve over all values of miss distance.

W is called the effective sweep width of the detector. This development eliminates the need to know the relative trajectory in order to predict the probability distribution of searching time until a detection occurs (the detection function).

Since the assumptions needed to derive the "formula of random search" are very restrictive, an assessment of the possible errors in (2) as an approximation for real detection processes is needed. Consider a definite range law detector systematically searching for a fixed target in a long strip of width twice the range (R_m) of the detector. Assume that the target is located at random in this strip. If the detector starts at the middle of one end of the strip and travels at constant speed through the center of the strip, the probability of detection is given by

$$P(L) = \begin{cases} \frac{WL}{A} & \text{for} \quad 0 \leq L \leq \frac{A}{W} \\ 1 & \text{for} \quad \frac{A}{W} < L, \end{cases} \qquad (3)$$

where A is the area of the strip. (End effects have been neglected in equation 3. Figure 1.3 shows the corresponding "law of random search" and systematic strip search detection functions. The slopes of these detection functions are the same at $L = 0$. The maximum difference between these two detection functions is $e^{-1} \cong .368$. But this value occurs only at $L = \frac{A}{W}$ which corresponds to the end of the systematic search of the strip. A difference of this magnitude occurs because because the systematic search derives the maximum possible amount of information

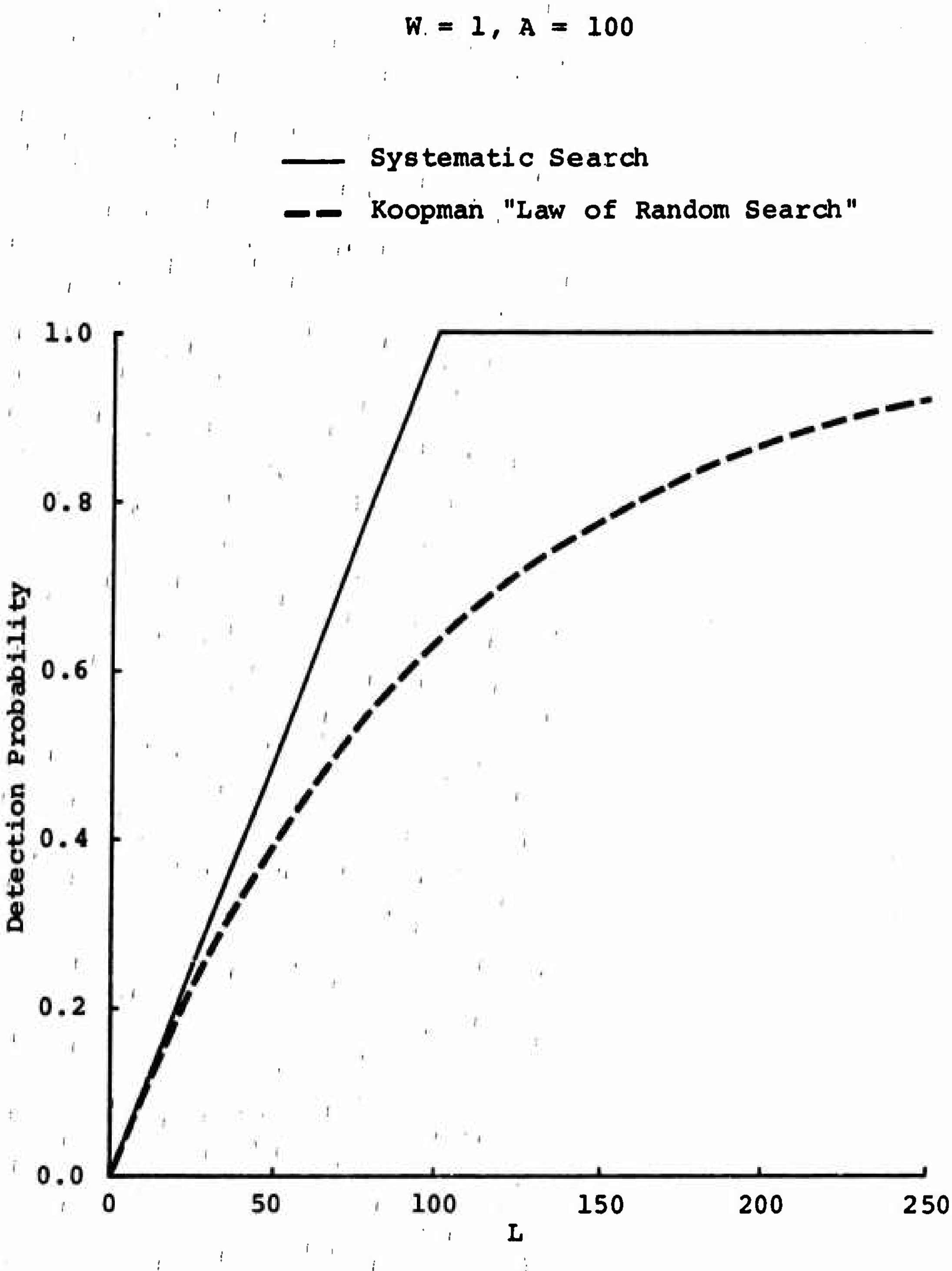

Figure 1.3 Detection Functions for Koopman "Law of Random Search" and Systematic Search

from the negative search result early in the search while the random search derives no information at all from a negative result. If it is possible for the detector to overlook the target or for the target to move from the unsearched portion of the area to the searched portion without being detected, the detection function will be closer to the "formula of random search" than is the systematic search detection function. As long as the target cannot evade the detector, the detection function should be bounded from below by the "law of random search." Thus, many real search situations may be expected to be governed by detection functions which are approximately of the "law of random search" exponential form.

Considering both a continuous search space and detector, Koopman also addressed the problem of allocating a limited amount of search "effort" to maximize the probability of detecting a fixed target whose location is described by a non-uniform probability distribution. Let

A = region containing the target,

x = a point in A,

$p(x)$ = probability density function for target location,

$w(x)$ = search effort density allocated to point x,

W = total amount of search effort available.

Assume that

$$\min_A p(x) = p_0 > 0,$$

and

Pr(Detect target | Target is at x) = 1 - exp(-w(x)).

Then, the problem is to

$$\max_w P[w] = \max_w \int_A p(x)[1 - \exp(-w(x))]dx,$$

Subject to

$$w(x) \geq 0, \qquad \int_A w(x)\, dx = W$$

Using variational arguments Koopman showed that

$$w^*(x) = \begin{cases} \ln p(x) + \frac{1}{A^*}\left[W - \int_{A^*} \ln p(x)dx\right] & \text{for } x \in A^* \\ 0 & \text{for } x \notin A^* \end{cases}$$

where

$$A^* = \left\{x \mid p(x) \geq b,\right.$$

$$\left. \ln b \int_{p(x) \geq b} p(x)\, dx = \int_{p(x) \geq b} \ln p(x)\, dx - w\right\} .$$

He noted that a series of optimal increments of effort $W_1, W_2, \ldots$ each based on the Bayesian updated target probability distribution, given that no detection has occurred, results in an optimal allocation of the total search $W_1 + W_2 + \cdots$.

The Koopman search optimization model is not easily related to the Koopman descriptive search model. If the underlying detection process is that of the descriptive model, then the "search effort density" of the optimization model is

$$w(x) = \int_{y_0}^{t_1} \gamma(x, \xi(t))\, dt,$$

where

(t_0, t_1) = time interval for the search,

$\xi(t)$ = position of the searcher at time t, and

$\gamma(x, \xi(t))$ = conditional detection rate at time t, given that the target is located at x.

The obvious, natural optimization problem is to maximize P[w] by selecting $\xi(t)$ subject to searcher mobility constraints. In general, this problem is much more difficult to solve than the Koopman optimization model in which the decision function, w(x), is not encumbered by the obvious consistency relations described here.

Since the Koopman search optimization model leads to a neat, non-trivial solution, the simplification obtained by ignoring consistency constraints on w(x) is not to be dismissed lightly as unreal. We wish to describe conditions under which this simplification is logically sound.

First, if the target location probability distribution is uniform over an area having dimensions large compared to the range of the detector and the searcher moves with constant speed at random, the simplification is valid. But this case assumes the optimization result as well as a very special target location probability distribution. For an arbitrary target location probability distribution the consistency constraints on w(x) are of two types: mobility and area coverage. The mobility constraints come from limitations in the speed and acceleration of the searcher. The area coverage constraint arises because at any given time the searcher is searching an area with the search rate varying over the area as prescribed by the detection rate function, γ. Thus, it is not possible to concentrate the search effort density arbitrarily. If the range of the detector is very small (compared to the distances between points for which p(x) are significantly different), the area coverage constraint will not affect the solution. Further, if the amount of search time is very large (compared to the time needed to search the entire area), the mobility constraints will not affect the solution. If the Koopman optimization simplification is to be valid in the limit for arbitrary target location probability distributions and amounts of search effort, the range of the detector must approach zero and the

mobility characteristics (speed and acceleration) must increase without bound. Therefore, the Koopman optimization model inherently assumes a pointwise detector with infinite mobility.

deGuenin (1961), Zahl (1963) and Arkin (1964) have generalized and embellished the Koopman continuous search space, continuous detector search optimization model to consider more general detection functions and problem constraints. But, these generalizations retain the basic pointwise detector characteristic. Considering the amount of search effort as a parameter, Arkin proved that, given an arbitrary continuous increasing detection function, there always exists a search plan which maximizes the probability of detecting the target for all amounts of search effort. Such a plan then, minimizes the expected effort needed to detect the target, assuming that no limit exists for the amount of search effort.

Blachman and Proschan (1959), Pollock (1960), Matula (1964), Chew (1967), Kadane (1968) and Ross (1969) analyzed discrete search space, discrete detector search models similar to the continuous search space, continuous detector models discussed above. The target in these models is located in one of a finite number of possible locations or "boxes." The detection process consists of a series of "looks" directed at individual boxes. On any given look at the box containing the target there is

a non-zero "overlook" probability (probability of not detecting the target). Pollock discovered that the search plan which minimizes the expected number of looks needed to detect the target also maximizes the probability of detecting the target for any given number of looks, given that there are two boxes and that the overlook probabilities are independent of the number of looks. Chew extended this result to the corresponding case with an arbitrary number of boxes. Considering a model with the cost of looking and the reward for detecting the target dependent on the box being searched, Ross found that the optimal search plans failed to substantiate three intuitive conjectures.

Charnes and Cooper (1958) considered a discrete search space, continuous detector search model which is a discrete search space analog of the Koopman search optimization model. The target is in one of several boxes, but the search process follows Koopman's "law of random search" exponential time to detect form. The problem of allocating a limited amount of search time to maximize the probability of detection is reduced to a separable, concave mathematical programming problem. The authors characterized the solutions by applying the Kuhn-Tucker conditions to the problem. Moore (1971) developed a slight generalization of this model and referred to it repeatedly as the "Standard Koopman Allocation" model.

This generalization of the Charnes and Cooper model is developed in Chapter 2 of this research and is referred to as the Koopman model.

While the (Charnes and Cooper) discrete search space model is formally identical to the analagous Koopman continuous search space search optimization model, these two models contain an essential difference in assumptions regarding the underlying detection process. The discrete search space version embodies the detection-at-a-distance property which characterizes many detectors such as radar, sonar, infrared detectors, etc. That is, if each box is itself as relatively large uniform area, the Koopman descriptive model leads one to expect the detection function to follow the "law of random search" approximately.

In an experimental study Stollmack (1968) empirically determined distributions of times for stationary experienced army personnel to visually locate a tank in the terrain surrounding Fort Knox, Kentucky. These distributions were found to agree well with Koopman's "law of random search." A wide range of empirical detection rates was observed depending strongly on the background as well as the distance between observer (detector) and tank (target). Thus, even though the processes are markedly different (experimental stationary visual detection of a stationary target and moving random search)

we have evidence of the appropriateness of an exponential detection function model with detection rates significantly dependent on the location.

Dobbie (1963) investigated the generality of the Koopman result concerning the overall optimality of successive optimal allocations of increments of effort to maximize detection probability. He found that such incremental allocations result in an optimal allocation of the total effort if the optimal effort allocation is non-decreasing in the variable available effort. For stationary targets this holds if the detection function is zero at zero effort density and is an increasing, concave function of effort density. Of course, Arkin's result above for continuous search space models establishes that the concavity restriction is not necessary. But, Arkin's result does not hold[1] for the discrete

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[1]This can be seen by considering a two-box example with uniform target location probability distribution. Let the detection functions for the two boxes be

$$f_1(t_1) = \begin{cases} t_1 & 0 \leq t_1 \leq 1 \\ 1 & 1 < t_1 \end{cases},$$

$$f_2(t_2) = \begin{cases} 0 & 0 \leq t_2 < \frac{1}{2} \\ 1 & \frac{1}{2} \leq t_2 \end{cases}.$$

Then, for amounts of search time, T, in the interval $(0,\frac{1}{2})$ the detection probability is maximized by $t_1=T$, but for T in the interval $[\frac{1}{2},\frac{3}{2}]$ the detection probability is maximized by $t_1=T-\frac{1}{2}$, $t_2=\frac{1}{2}$.

search space models. Dobbie also argued that maximizing the detection probability for a given amount of search effort may not correspond to the searcher's real motivation. He suggested that search objective functions should be carefully selected on the basis of the characteristics of the search problem context.

Novosad (1961) and Mela (1961) apparently independently published search models showing that maximizing information gain (change in entropy) can lead to different search allocations than those which maximize detection probability. Then, Danskin (1962) published two papers in which information gain was taken as the search objective. Tognetti (1968) and Kadane (1971) considered discrete search space, discrete detector models in which the objective is to maximize the probability of either detecting the target or correctly guessing the target location following the search.

Smith (1969) first introduced false detections explicitly in a discrete search space, discrete detector model similar to those of Pollock and Chew. That is, in addition to the overlook probability, he assumed a nonzero probability of "detection" on a look when, in fact, the target is not present in the box being searched. Using Bayes' formula he derived the sequential search procedure which maximizes the probability that the first

"detection" is a true detection. This optimal procedure consists of always allocating the next look to a box having the highest Bayesian updated probability of containing the target. This optimal search is shown to minimize the expected number of looks until a "detection" occurs.

Stone and Stanshine (1971) considered a continuous search space, continuous detection process search model involving false detections. The false detections are assumed to result from the detection of false targets which cannot be distinguished from the real target by the basic search system. These false targets can, however, be identified by interrupting the basic search and employing special methods to identify the contact as either target or false target. In terms of our taxonomy this contact investigation may be regarded as the response process. The model is characterized by a repeated, continuous activity response process coupled with the (basic) search process by some interference mechanism which requires that the basic search cease while a contact is being investigated. In this model the target location is characterized by the probability density function, $f(x)$. The false targets are described by their collective "false target density function," $\delta(x)$, which expresses the number and location probability distribution of the false targets.

The contact process is governed by a contact function analogous to the detection function of the Koopman model. That is if either the target or a false target is located at point x and effort density $m(x,s)$ is expended at x, the probability of contacting the target or false target is $b(m(x,s))$. Assume that

(i) $b(0) = 0, \quad \lim_{z\to\infty} b(z) = 1,$

(ii) b', the derivative of b, is a continuous positive, decreasing function

Let

$m(x,s)$ = search effort density allocated to point x when s units of basic search time have been expended.

The constraints on m are

(i) $m(x,s) \geq 0,$

(ii) m is non-decreasing in the second argument for each x, and

(iii) $\int m(x,s)\, dx = Us.$

The probability of contacting the target by basic search time s is

$$P(m,s) = \int f(x)\, b(m(x,s))\, dx.$$

The time needed to investigate a false target is assumed to be a random variable, $T(x)$, which depends on the location of the false target.

$$\xi(s) = \int \delta(x)\, b(m(x,s))\, dx$$

is the expected number of false contacts by basic search time s. If all contacts are investigated immediately, the expected time spend in identifying false contacts before basic search time s is

$$T(x)\delta(x)b(m(x,s))\ dx.$$

Immediate contact investigation is shown to be optimal if the contact investigation phase cannot be interrupted to switch back to the basic search phase. It is shown that no search-identification plan can maximize the probability of contacting and identifying the target before every amount of total search and identification time. A Neyman-Pearson type of allocation is shown to minimize the expected total search and identification time needed to locate the target. The resulting optimal allocation pattern for a simple example is shown to be closely related to the corresponding Koopman optimal allocation model with no false targets.

This false detection model involves a curious mixture of a fixed basic search plan which is never altered in response to the outcome of the search and opportunistic exploration of the contacts which occur. By adhering to this fixed basic search plan the model avoids detailed consideration of the source of false targets and the effect of the identification of false targets on the underlying probability distribution structure. 'hat is, the

model lacks sufficient definition to permit the computation and use of false probability distributions of the number and locations of uncontacted false targets via Bayes' formula.

Pollock (1971) considered the nature of the overall problem setting of search theory. He suggested that search models have too often oversimplified or ignored the detection theory and decision theory aspects of the problems being addressed. Search-related detection and decision models involving significant search modeling are similarly lacking in the literature. He argued that separate modeling of these three parts of search processes has lead to interface problems. The difficulties encountered by this fragmented modeling appear to constitute an excellent example of the suboptimization syndrome pointed to by Hitch and McKean (1960).

Compared to the apparent complexity and variety of practical search problems as reflected in the taxonomy of Section 1.1, current search theory models are distinguished by their simplicity. The search literature is dominated by models incorporating essentially generalizations of Koopman's original search optimization model. A few authors have noted that the detection probability maximization of these models may be misleading as a guide for allocating search resources in many search problems. Yet,

only recently have researchers begun to consider search problems from a more comprehensive point of view and to investigate other possible objective functions for search optimization.

1.3 Area of Research

In Section 1.1 we presented a rather complicated taxonomy for classifying search-response models. This taxonomy represents an intuitive conjecture of which distinctions are fundamental to the analysis of the vast variety of practical problems involving search-response processes. Of paramount importance in this taxonomy is the notion that the search results represent intermediate states in the overall system operation rather than the essential motivation for conducting search activities. Thus, we regard the response process as the primary activity of interest to the searcher.

The importance and effect of the response process can be considered in context of many combinations of model characteristics noted in the taxonomy of Section 1.1. Although it is felt that analysis of many of these structures will be necessary to fully understand the relationship between response and search processes of necessity this research has focussed on but a few dimensions to develop some preliminary understanding of their dependency.

Because, a priori, we expect the analysis of search-response problems to depend strongly on false detection effects, noisy detectors are examined. The analysis is performed in conjunction with the principle discussion of scheduled and unscheduled response processes. Discrete search space, continuous detector search models similar to that analyzed by Charnes and Cooper are used in the analysis.

Detection events are inherently decisions. Therefore, the "detection" rate can be changed by changing the criterion used for making a positive detection decision. But, changing the criterion for the detection decision changes the false detection rate as well as the overall detection rate. Thus, there exists a technical tradeoff relationship between the detection (decision) rate and the false detection rate. This relationship is called the receiver operating characteristic curve. An important practical question in the design and operation of detectors is the selection of an operating point on the receiver operating characteristic curve. Although our models can be used in analyzing this operating point selection problem we have concentrated attention on analyzing the search allocation problems with the operating point assumed fixed.

As previously noted, the purpose of this research is to (a) develop structures which link the search and re-

sponse processes and (b) use these to examine the effect that the response process has on classical search strategies and gain some physical insight into the relationship between the two processes. Hypothetical search-response system models with false detections are developed and compared with corresponding classical search optimization models. Such comparisons provide guidelines regarding the robustness of the classical models as guides for search decision makers and develop insight regarding the importance of the response process and false-detection model elements. A modified version of the classical Koopman model is used in these comparisons since most efforts in search theory have been essentially embellishments of it, and accordingly, it is felt that results would apply to other more sophisticated search optimization models.

Emphasis in the research is on the relationship between model assumptions and results. Therefore, tedious technical detailed developments are presented in appendices rather than in the main body of the report. Chapter 2 considers an unscheduled response process search model. Chapter 3 considers three closely related scheduled response process search models. Chapter 4 summarizes the major results of the analyses and discusses directions for future research.

CHAPTER 2

SEARCH MODELS WITH UNSCHEDULED RESPONSE PROCESSES

In this chapter we develop and analyze two models of unscheduled response-search processes. The first model is the discrete search space version of Koopman's original search allocation model. Hence, we call this model the Koopman model. As previously noted, it has been throughly studied and embellished by other researchers. We include the Koopman model as the primary model examined in the literature with which succeeding models are to be compared. The other model represents a simple extension [1] of the Koopman model including explicit treatment of the response process and false detection effects.

The models of this chapter consider only two possible target locations or "boxes." The generalization of these models to consider an arbitrary number of boxes has been performed and follows directly from the two box case. There were two main reasons for limiting this discussion to the two boxes version of these models:

1. The models of Chapter 3 with which we wish to

[1] We have developed several models similar to this extension of the Koopman model. The specific one included in the analysis illustrates the basic character of these models and their implications.

compare results are not easily generalized to the corresponding models considering arbitrary numbers of boxes.

2. The two boxes versions embody the significant conceptual results we wish to illustrate. Thus, the symbolism and graphical complications associated with discussing the multiple boxes versions simply detract from the effects we want to emphasize.

2.1 The Koopman Search Model

Exactly one stationary target is located in one of two boxes with

$$p_i = \Pr(\text{Target is in box } i).$$

The detection process for searching in the box which contains the target follows Koopman's formula of random search. That is, the conditional search time-to-detection, given that the target is in box i, is characterized by[1]

$$e^{-k_i t} = \Pr(\text{Target is not detected} \mid \text{Target is in box } i),$$

following a search of duration t in box i. The searcher allocates a limited amount of available searching time,

---

[1]Use of an exponential detection law in this analysis is based in part on the functions mathematical simplicity which facilitates interpretation of the results and its origin, occurring axiomatically in random search processes and empirically in visual detection processes (Stollmack, 1968).

T, between the two boxes to maximize the probability of detecting the target.

Let

T_i = amount of available time allocated to searching box i.

Then the Koopman search allocation problem is to select T_1 and T_2 which

$$\text{Max DP} = \text{Max} \sum_{i=1}^{2} p_i(1-e^{-k_i T_i}) \tag{4}$$

subject to $T_1, T_2 \geq 0$ and $T_1 + T_2 \leq T$.

Note that p_1, p_2, k_1, k_2 and T are necessarily non-negative. If any of these parameters are zero, the problem is trivial. Therefore, we assume that these parameters are all positive.

Appendix A uses the Kuhn-Tucker conditions to develop a solution algorithm for a generalization of this Koopman search allocation problem. Applied to the Koopman problem, (4), this algorithm yields the following: Let indices be assigned such that $p_1k_1 \geq p_2k_2$. Also, let (T_1^*, T_2^*) denote an optimal allocation and DP* the corresponding target detection probability.

1. (T_1^*,T_2^*) is unique with $T_1^* + T_2^* = T$ for any $0 < T < \infty$.

2. The solution can be expressed analytically in terms of the problem parameters and

$$W = \frac{1}{k_1} \ln \frac{p_1 k_1}{p_2 k_2} \geq 0.$$

i. If $0 < T \leq W$,

$$(T_1^*, T_2^*) = (T, 0).$$

ii. If $W < T$,

$$T_1^* = W + \frac{k_2}{k_1 + k_2}(T - W) \text{ and}$$

$$T_2^* = \frac{k_1}{k_1 + k_2}(T - W).$$

3. The corresponding optimal target detection probability is

$$DP^* = \begin{cases} p_1\left(1 - e^{-k_1 T}\right) & \text{for } 0 < T \leq W \\ 1 - p_2\left(1 + \frac{k_2}{k_1}\right)\exp\left[\frac{W-T}{\frac{1}{k_1} + \frac{1}{k_2}}\right] & \text{for } W < T. \end{cases}$$

The trace of an optimal allocation plan, (T_1^*, T_2^*) as a function of T, is shown in Figure 2.1a. We will call such traces optimal trajectories. Koopman model optimal trajectories are piecewise linear, monotonic non-decreasing in form. For small amounts of available searching time, the optimal search is concentrated entirely in box 1.

$p_1 = 0.5,\ k_1 = 0.10$

$p_2 = 0.5,\ k_2 = 0.05$

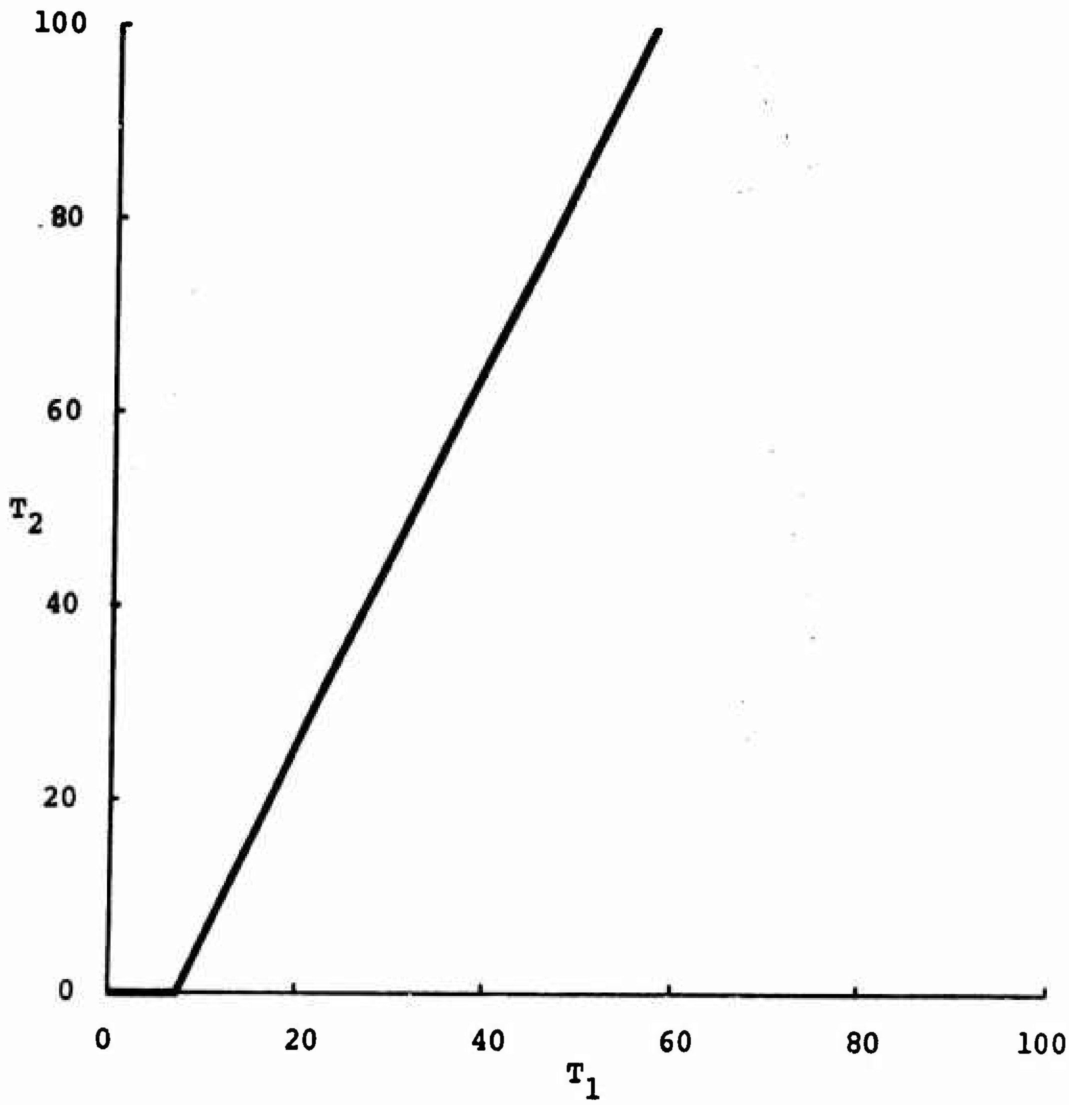

Figure 2.1a Optimal Koopman Search Plan

For larger amounts of available searching time, the allocation is shared with constant marginal allocations of available searching time to each of the two boxes. These marginal allocations depend on the detection rates but not on the prior target location probability distribution.

The non-decreasing character of the optimal trajectories implies that the search decision maker need not know the amount of available searching time, T, in order to search optimally. Therefore, the Koopman model optimal search allocations are also good search plans for the corresponding problems in which the scarcity of search effort enters through a search cost function which increases as the search progresses. Thus, the Koopman search plan minimizes the expected cost of detecting the target.[1]

Figure 2.1b shows the optimal detection probability, DP*, as a function of available searching time, T, for the optimal search shown in Figure 2.1a. This optimal detection probability function consists of portions of two exponential "charging" functions: one for $0 < T \leq W$ and the other for $W < T$. The slopes of these two sections of DP* are the same at W so that the marginal return is continuously decreasing over the entire interval $(0, \infty)$.

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[1]Dobbie (1963) proved that non-decreasing search allocation plans to maximize detection probability also minimize the expected time to detect the target. The extension of this result to the case of arbitrarily increasing cost as a function of search time is trivial.

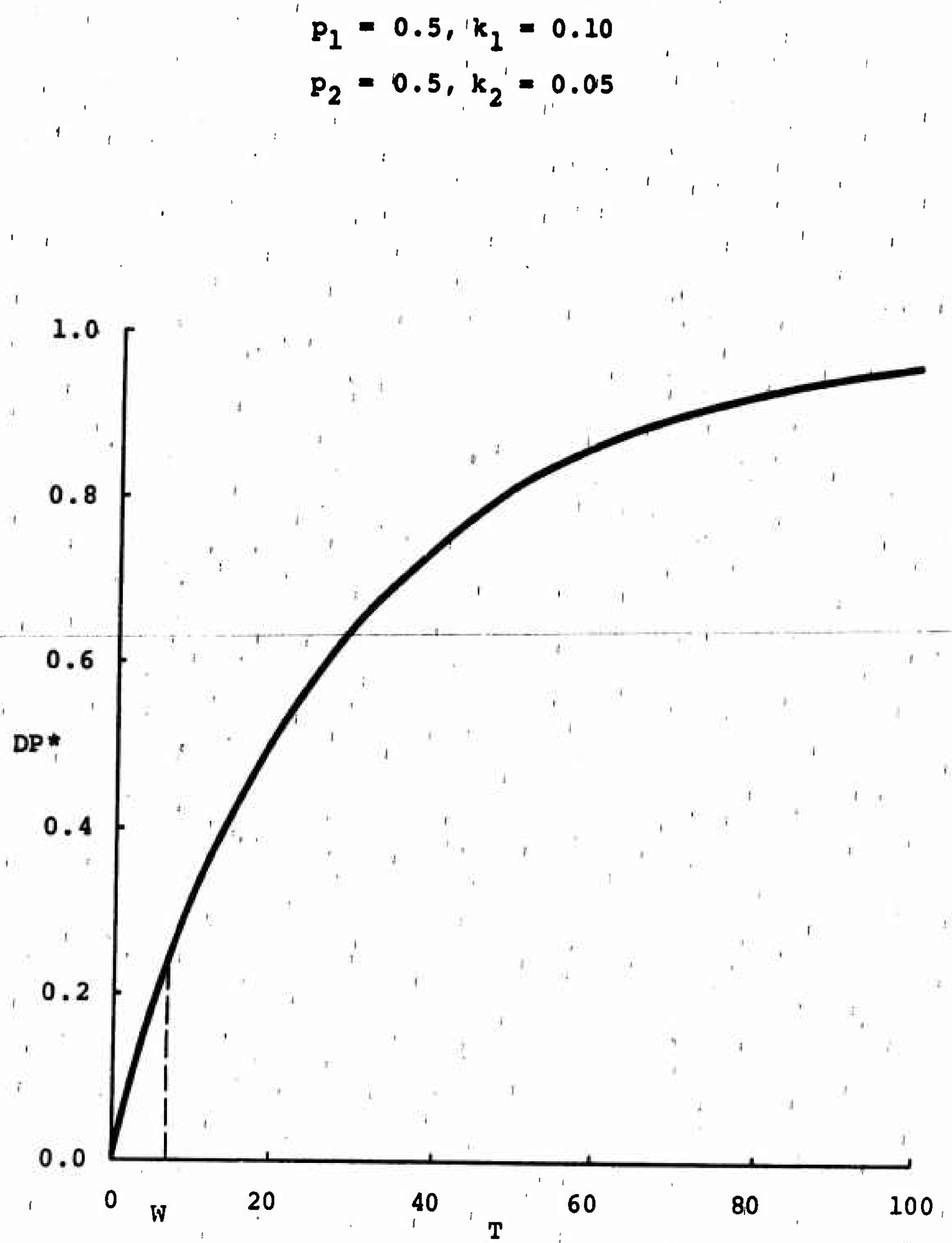

Figure 2.1b Koopman Model Optimal Detection Probability vs Available Search Time

2.2 *An Unscheduled Response (UR) Search Model*

In this section we analyze a search model involving false detections and an unscheduled response process. The kinds of real problem situations which motivate this UR search model are illustrated by the following scenario: In a ground war one force sends a scout patrol to search for an enemy combat unit. The enemy unit maintains a guard detail which searches for scout patrols. When either the guard detail or the scout patrol detects the other, an engagement between these two units ensues with surprise advantage for the unit which made the detection. Assume that if an adversary is present any erroneous attempt to initiate an engagement (when the adversary has not, in fact, been detected) exposes the "attacker" to certain detection.

The key elements in this problem are:

i. The primary interest of the searcher is in the response process which ensues if and only if a detection occurs.

ii. The detection system is necessarily imperfect so that false detections may occur.

We want to investigate the effects of these characteristics of search problems on the optimal search allocations. Therefore, we consider a specific search model having these characteristics.

Exactly one stationary target is located in one of two boxes with

$$p_i = \Pr(\text{Target is in box } i).$$

The detection system produces "contacts" or positive decisions that the target has been found. These are characterized by random times-to-contact with

$$e^{-k_i t} = \Pr(\text{No contact in box } i \mid \text{Target is in box } i)$$

for a search duration t in box i. Contact events may be of two types: "false contacts" and "true detections." False contacts lead to unfavorable expected response process results while true detections lead to favorable expected response process results. The false contacts occur independently of the search times-to-contact with

$$\beta_i = \Pr(\text{False contact} \mid \text{Contact in box } i,\ \text{Target is in box } i).$$

Any contact event terminates the search and initiates the response process. If the response is initiated as the result of a false contact, the searcher expects unfavorable results from the response process. The purpose of the search is to attempt to detect the target, initiating the response process under conditions favorable to the searcher. Since the utility of the response is different under these types of contacts we let

d_i = Expected utility from the response process given a true detection in box i,

f_i = Expected utility from the response process given a false contact in box i containing the target.

Assume that the response process so intimately involves

interactions between the searcher and the target that no response can occur in the box that does not contain the target. The expected utility parameters, d_i and f_i, are relative to the search-response outcome in which no response occurs. Additionally, in the scenario we let

Q_i = Pr(Patrol survives | Patrol initiates duel),

q_i = Pr(Patrol survives | Enemy unit initiates duel).

Suppose that the searcher's utility structure is

$$U = P_S - vP_E ,$$

where

U = searcher's utility

P_S = Pr(Patrol survives)

P_E = Pr(Enemy unit survives)

v = a parameter expressing the relative utility of patrol and enemy unit survival.

Applying the definitions of d_i and f_i to scenario 2,

$$d_i = Q_i - v(1-Q_i),$$

and

$$f_i = q_i - v(1-q_i).$$

We assume that false contacts by one searcher (scout patrol or guard detail) expose the erroneous "attacker" to certain counter-detection if and only if the two adversaries are in the same box. Suppose that both searchers operate exponential detections (Koopman "formula of random search" detectors) i.e.,

$e^{-r_i t}$ = Pr(Patrol fails to contact | Patrol and enemy are in box i for t time units)

$e^{-r_i' t}$ = Pr(Enemy fails to contact | Patrol and enemy are in box i for t time units),

γ_i = Pr(False contact | Contact by patrol in box i),

γ_i' = Pr(False contact | Contact by enemy in box i).

We wish to compute the corresponding overall contact rates, k_i, and false contact probabilities, β_i, as viewed by the scout patrol. There are two cases to consider.

Case 1 Suppose that the searching times-to-contact for the two searchers are independent. Then

$e^{-r_i t} \cdot e^{-r_i' t}$ = Pr(No contact by either | Patrol and enemy search for t units in box i).

The joint contact rate is simply the sum of the two individual contact rates, i.e.,

$$k_i = r_i + r_i'.$$

The false contact parameters, β_i, are the probabilities that the enemy will initiate the real duel following a contact in box i given that the enemy is in box i. To compute this let us first use Bayes' formula to compute

Pr(Contact by patrol | Enemy is in box i, A contact occurs in box i)

$$= \lim_{dt \to 0^+} \frac{e^{-k_i t} r_i dt(1-r_i' dt)}{e^{-k_i t}[r_i dt(1-r_i' dt)+r_i' dt(1-r_i dt)]} = \frac{r_i}{k_i}. \qquad (5)$$

Then,

$$\beta_i = \frac{r_i}{k_i}\gamma_i + \frac{r_i'}{k_i}(1-\gamma_i'). \tag{6}$$

That is, the false contacts of the model come from two sources: erroneous interpretations of detector data by the patrol and true detections by the enemy.

Case 2 Suppose that the searching times-to-contact for the two searchers are not independent. Then, to compute the joint contact rate one must explicitly treat the dependency that is involved. The joint contact rate is not necessarily the sum of the contact rates of the two searchers. And, neither is the probability that the first contact is made by the first searcher necessarily given by (5). That is, determining the k_i and β_i parameters for a two-sided correlated search requires modeling of the physical details of the search situation involved.

The UR model of this chapter was first developed incorporating the independent two-sided search assumption of Case 1 above. This assumption was removed from the main development because we think that the dependency involved in many, if not most, real two-sided search situations is significant. We assert that a significant positive correlation should be expected for two searchers randomly searching a large area using range dependent detectors with the detector ranges small compared to the

dimensions of the area.

The reason for this expected correlation rests on the fact that for most of the searching time in any particular realization of the random search, neither searcher can possibly detect the other because they are separated by a distance greater than the range of their detectors. Consider the extreme example in which the two searchers operate identical definite range law detectors which instantaneously detect with probability one any target which comes within the range of the detectors. Koopman has shown that the exponential law of random search is a reasonable approximation to the probability distribution of times-to-detect for such a detector as either the detector or target, or both, move at constant speed more or less randomly in the search area spending approximately the same amount of time in sub-search areas of the same size. But, in the two sided version of this search the two searchers always detect one another at the same time. The joint contact rate is the same as the contact rate of either searcher -- not the sum of the contact rates of the two searchers. And the race to contact the other searcher before one is detected always ends in a draw, an outcome which occurs with probability zero for the corresponding independent two-sided search.

While the definite range law example cited is extreme,

the effect of similar range dependence of the detector capabilities can be expected to lead to a positive correlation between the times-to-contact for the two searchers, joint contact rate less than the sum of the individual searchers' contact rates, and some relationship of unknown form in place of (6).

If the times-to-contact for the two searchers are approximately independent, the joint contact rate and probability that any response is unfavorable are easily determined in terms of possible experimental data for the individual detectors. But, if the two contact processes are not approximately independent, one must develop theory appropriate to the situation being considered to determine the joint contact rate and probability that any response is unfavorable from the individual detector's characteristics.

In the search allocation problem the search decision maker has a limited amount of available searching time, T, to allocate between the two boxes to maximize the expected utility from the response process. If the target is in box i, the expected utility is

$$U^i = \left[1-e^{-k_i T_i}\right]\left[(1-\beta_i)d_i + \beta_i f_i\right] ,$$

where

T_i = amount of time allocated to searching box i.

Therefore, the search decision maker's allocation problem can be expressed as

$$\text{Maximuze } U = \text{Max} \sum_{i=1}^{2} B_i(1-e^{-k_iT_i})$$

$$\text{subject to} \quad T_1, T_2 \geq 0, \quad T_1 + T_2 \leq T, \tag{7}$$

where

$$B_i = \left[(1-\beta_i)d_i + \beta_i f_i\right]p_i.$$

2.2.1 Comparison of UR and Koopman Models

The decision maker's allocation problem in the UR model is identical in form to the Koopman model allocation problem. The B_i parameters of the UR model replace the corresponding prior target location probabilities, p_i, of the Koopman model. But, while the target location probabilities are non-negative, the B_i parameters may be negative as well as positive or zero. If $B_i < 0$, clearly $T_i^* = 0$ in any optimal solution. And if $B_i = 0$, the objective function is independent of T_i so $T_i^* = 0$ is optimal for any T. These immediate results may be used to modify the solution of the Koopman allocation problem to obtain the solution for the UR allocation problem. Assigning indices such that $B_1k_1 \geq B_2k_2$ we have:

1. $B_i < 0$ implies that $T_i^* = 0$.
2. If there exists i such that $B_i > 0$, $T_1^* + T_2^* = T$.
   If $B_1 \leq 0$ and $B_2 \leq 0$, $(T_1^*,T_2^*) = (0,0)$ is optimal.
3. If $B_1 > 0$ amd $B_2 \leq 0$, $(T_1^*,T_2^*) = (T,0)$.
4. If $B_2 > 0$, there exists

$$W = \frac{1}{k_1} \ln \frac{B_1 k_1}{B_2 k_2} \geq 0 \tag{8}$$

such that:

i. If $T \leq W$, $(T_1^*,T_2^*) = (T,0)$.

ii. If $W < T$,

$$T_1^* = W + \frac{k_2}{k_1+k_2}(T-W) \quad \text{and}$$

$$T_2^* = \frac{k_1}{k_1+k_2}(T-W).$$

The optimal objective function is

$$U^* = \begin{cases} B_1\left[1 - e^{-k_1 T}\right] & \text{for } 0 < T \leq W \\ B_1+B_2\left[1-\left(1+\frac{k_2}{k_1}\right)\exp\left\{\frac{W-T}{\frac{1}{k_1}+\frac{1}{k_2}}\right\}\right] & \text{for } W < T. \end{cases}$$

The solution of the UR search allocation problem is very similar to that for the related Koopman allocation problem. The expected utilities given that a contact is made in the correct box combine simply with the prior probabilities to yield the relevant B_i values. Each B_i value may be interpreted as the expected utility associated with an unlimited search in box i. If one considers the Koopman allocation problem as a simple expected utility model with the expected utility of finding the target the same for the two boxes, the prior probabilities, p_i, can also be interpreted as the expected utilities associated with unlimited searches in each of the two boxes. The major difference is that B_i values can be negative so that searching the box i may be harmful rather than useful.

The k_i for the UR problem, which play the same role in the solution as the detection rates for the Koopman model, are not real detection rates. Rather, these k_i represent contact rates including the false as well as the true detections. The true detection rates are $(1-\beta_i)k_i$. Except for this difference in meaning for the "detection" rates the dependence on detection rates is the same for the Koopman and UR problems.

The potential importance of the difference between the Koopman and UR models as guides for search decision makers is illustrated by the following comparison:

Suppose the real search problem corresponds to the UR model but the decision maker plans his search based on the corresponding Koopman model. Assume that the detection rates used in the Koopman model are the total contact rates, k_i, of the UR model. Table 2.1 lists the parameter values for this comparison of the two models.

Table 2.1

i	p_i	k_i	β_i	d_i	f_i	B_i
1	.5	1	.5	.2	-.6	-.10
2	.5	1	.1	.2	-.4	.07

Parameter Values for Koopman and UR Model Example Comparison

The optimal Koopman search plan is $(T_1^*, T_2^*) = (.5T, .5T)$. Since B_1 is negative and B_2 is positive, the optimal UR search plan is $(T_1^*, T_2^*) = (0, T)$ for all $0 < T < \infty$. Figure 2.2 shows the values of the UR objective function for these two search plans. The values of UR objective function attained by the Koopman search plan are always negative. That is, it is better not to search at all than to maximize the "detection probability." Thus, optimizing the wrong objective function, in this case, fails to lead to a reasonable first approximation to the real optimal allocation.

i	p_i	k_i	β_i	d_i	f_i	B_i
1	.5	1	.5	.2	-.6	-.10
2	.5	1	.1	.2	-.4	.07

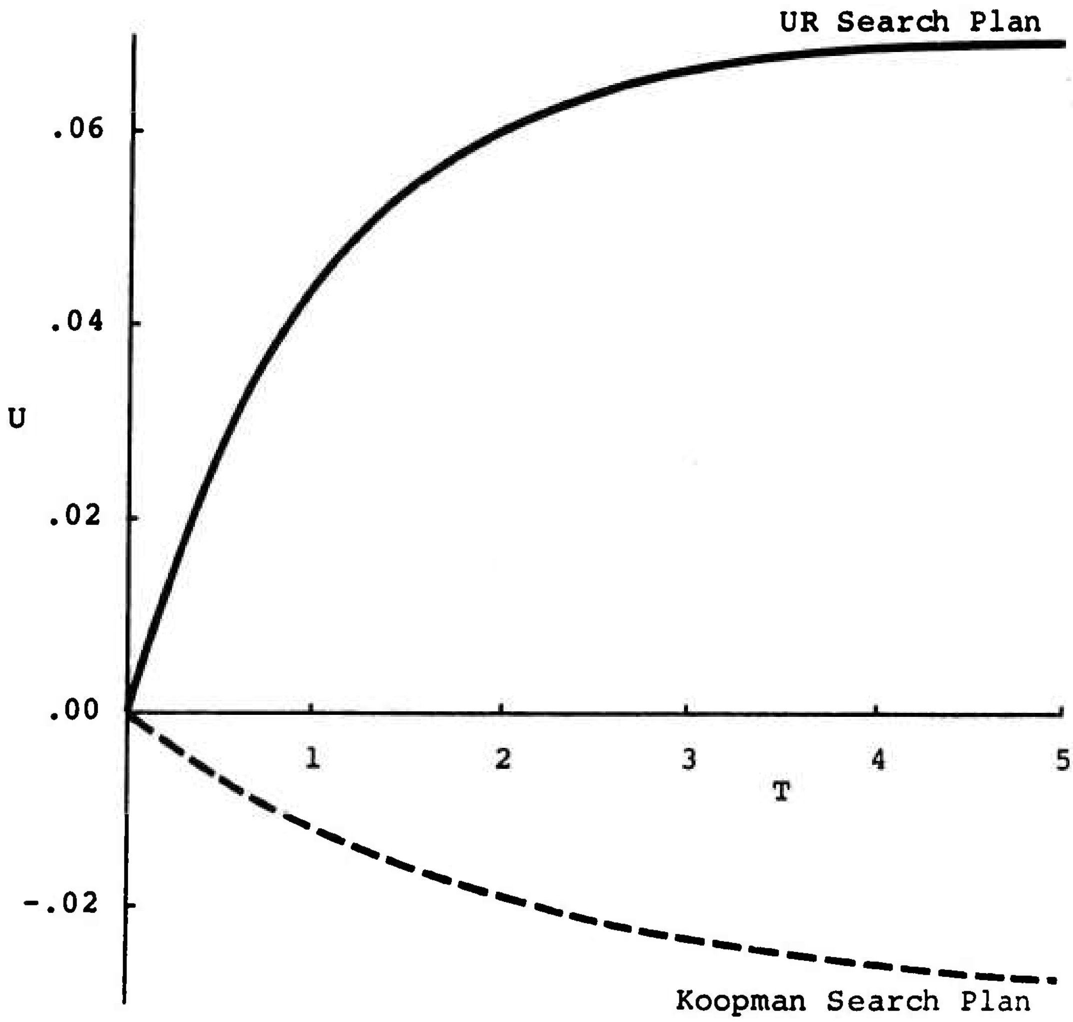

Figure 2.2 UR Objective Function for Koopman and Optimal Search Plans

Disasterous consequences such as those depicted by Figure 2.2 are not characteristic of the results of using the Koopman search plan for any UR search problem situation. If B_1 and B_2 are both positive, any allocation of large T with both $k_1T_1 >> 1$ and $k_2T_2 >> 1$ will yield an objective function value of approximately B_1+B_2. Thus, if searching both boxes is worthwhile for large T, the Koopman search plan is approximately optimal for large T. Consequently, except for cases in which the searcher has overlooked overriding danger, the Koopman search plan is significantly inferior to the optimal UR search plan only for small and moderate amounts of available searching time, T.

The Koopman search allocation may constitute a good search plan for small T as well. Consider a UR search problem with $\beta_1 = \beta_2$, $d_1 = d_2$ and $f_1 = f_2$. Then, the B_i parameters are proportional to the prior target location probabilities, p_i. Consequently, either no search is worthwhile (if $B_1, B_2 \leq 0$) or the Koopman search is optimal. Another relation among parameters which also leads to the Koopman search plan being optimal is $d_1 = d_2 = f_1 = f_2$. This condition may be expressed as: The expected response utility is the same for all contacts. For such problems the objective function is independent of the false detection parameters, β_1 and β_2.

2.2.2 *Sensitivity of UR Model Results*

The false detection effects enter the UR model formulation only through the B_i parameters, which represent expected utilities associated with unlimited searching in each of the two boxes. The B_i parameters (except when neither B_1 or B_2 is positive) affect the allocation in a significant way only for small T. With the contact rates, k_i, the B_i parameters determine the point at which the optimal trajectory departs from the T_1 axis. This point depends on the logarithm of the ratio B_1/B_2 as shown by (8). Thus the position of the optimal trajectory line for large T is relatively insensitive to the model parameters (assuming that $B_2 > 0$). The slope of the optimal trajectory line for large T depends only on the detection rates.

Consider the sensitivity of the objective function, U, to T_1 along the "budget" line $T_1 + T_2 = T$. Along this line

$$\frac{dU}{dT_1} = \frac{\partial U}{\partial T_1} - \frac{\partial U}{\partial T_2}$$

$$= B_1 k_1 e^{-k_1 T_1} - B_2 k_2 e^{-k_2 T_2} \quad .$$

Thus, if $k_1 T_1 >> 1$ and $k_2 T_2 >> 1$, dU/dT_1 is small even if

(T_1,T_2) is far from the optimal trajectory. Also, consider the marginal expected value, dU/dT, along a line parallel to the optimal trajectory line for large T. Along such a line

$$\frac{dU}{dT} = \frac{\partial U}{\partial T_1}\frac{dT_1}{dT} + \frac{\partial U}{\partial T_2}\frac{dT_2}{dT}$$

$$= C \exp\left\{-\frac{k_1 k_2}{k_1+k_2} T\right\} ,$$

for some positive, C.

Next let us consider the sensitivity of the upper bound for the expected utility as T grows without bound. Let

$$\bar{U} = \lim_{T\to\infty} U^*.$$

Clearly,

$$\bar{U} = \bar{U}_1 + \bar{U}_2 \tag{9a}$$

where

$$\bar{U}_i = \max\{0, B_i\} , \quad \text{for } i = 1,2. \tag{9b}$$

There exists a threshold value

$$\bar{\beta}_i = \frac{d_i}{d_i - f_i}$$

such that

(i) if $\beta_i \geq \bar{\beta}_i$, $B_i \leq 0$,

(ii) if $\beta_i < \bar{\beta}_i$, $B_i > 0$.

If f_i is non-negative, $\bar{\beta}_i \geq 1$ and, therefore, $B_i > 0$ for all $0 \leq \beta_i < 1$. If f_i is negative, $0 \leq \bar{\beta}_i < 1$ and, therefore $B_i > 0$ only for $0 \leq \beta_i < \bar{\beta}_i < 1$. Because $T_i^* = 0$ is optimal if $B_i \leq 0$, the results are insensitive to β_i for $\beta_i > \bar{\beta}_i$. Let us examine the sensitivity to β_i in the interval $0 < \beta_i < \min\{\bar{\beta}_i, 1\}$. In this interval

$$\frac{\partial \bar{U}}{\partial \beta_i} = \frac{\partial \bar{U}_i}{\partial \beta_i} = \frac{\partial B_i}{\partial \beta_i}$$
$$= -p_i(d_i - f_i) \tag{10}$$

That is, $\bar{U}$ is linearly decreasing in β_1 and β_2.

The sensitivity of $\bar{U}$ to β_1 and β_2 may be illustrated by plotting iso-$\bar{U}$ curves in the (β_1, β_2) plane. Figures 2.3a to 2.3d depict such iso-$\bar{U}$ contours for sets of parameters which illustrate the basic characteristics of the dependence of $\bar{U}$ on β_1 and β_2. From (7) and (9) we can

$p_1 = 0.2,\ d_1 = 1,\ f_1 = 0,\ \bar{\beta}_1 = 1$

$p_2 = 0.8,\ d_2 = 1,\ f_2 = 0,\ \bar{\beta}_2 = 1$

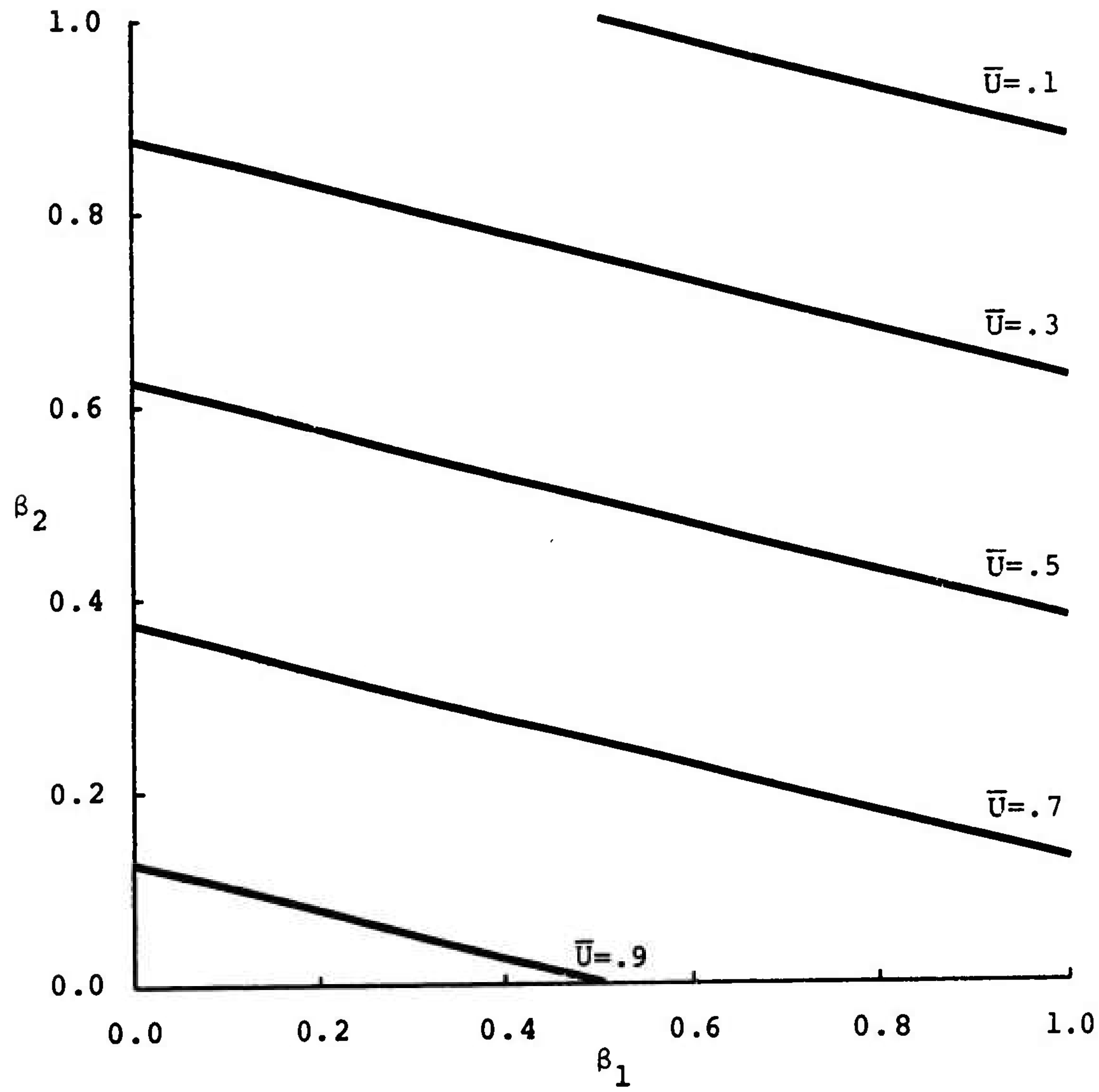

Figure 2.3a UR Model Iso-$\bar{U}$ Curves in (β_1, β_2) Plane

$p_1 = 0.2,\ d_1 = 1,\ f_1 = 0,\ \bar{\beta}_1 = 1$

$p_2 = 0.8,\ d_2 = 1,\ f_2 = -.5,\ \bar{\beta}_2 = 2/3$

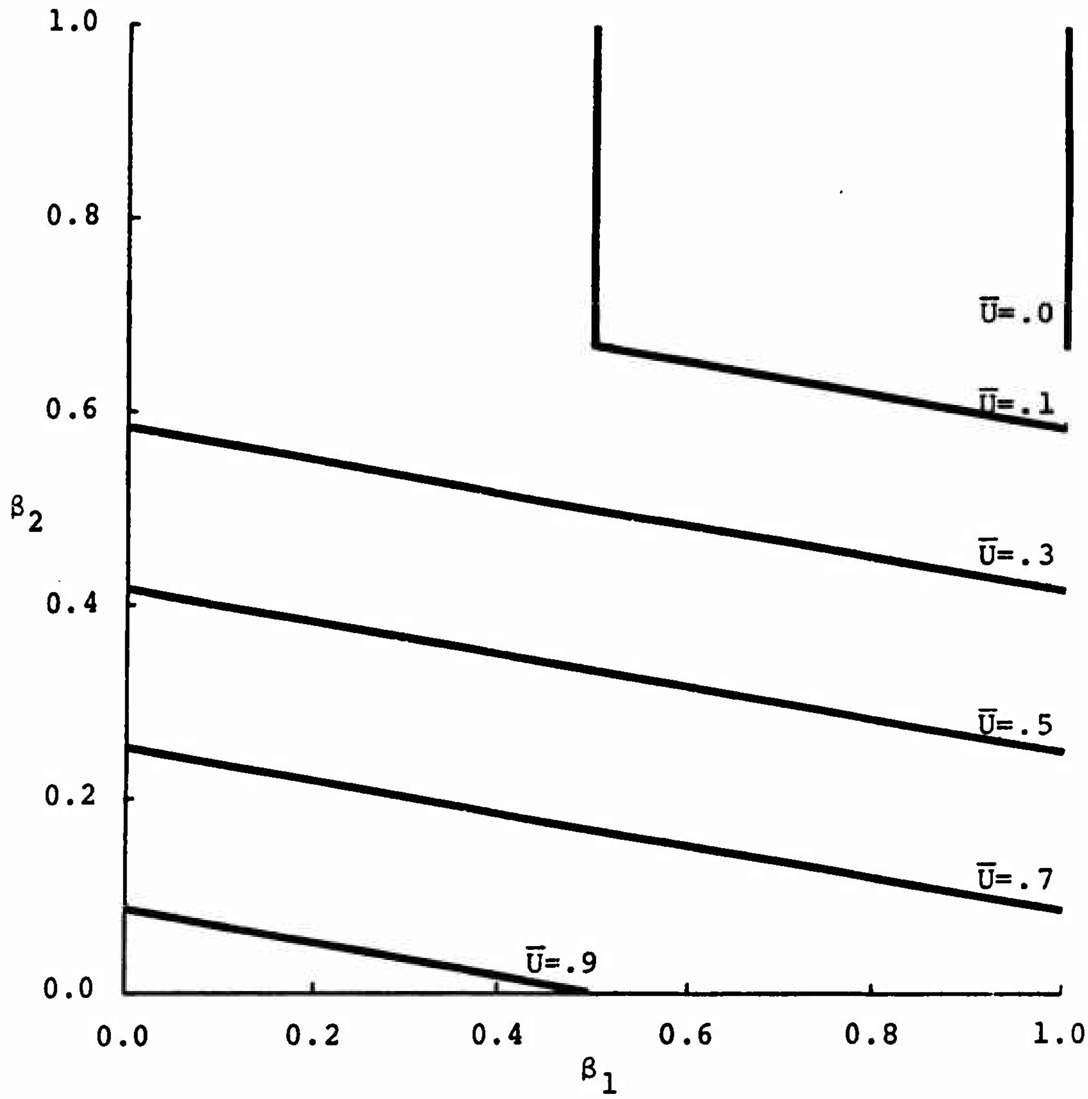

Figure 2.3b UR Model Iso-$\bar{U}$ Curves in (β_1, β_2) Plane

$p_1 = 0.2,\ d_1 = 1,\ f_1 = -.5,\ \bar{\beta}_1 = 2/3$

$p_2 = 0.8,\ d_2 = 1,\ f_2 = 0,\ \bar{\beta}_2 = 0$

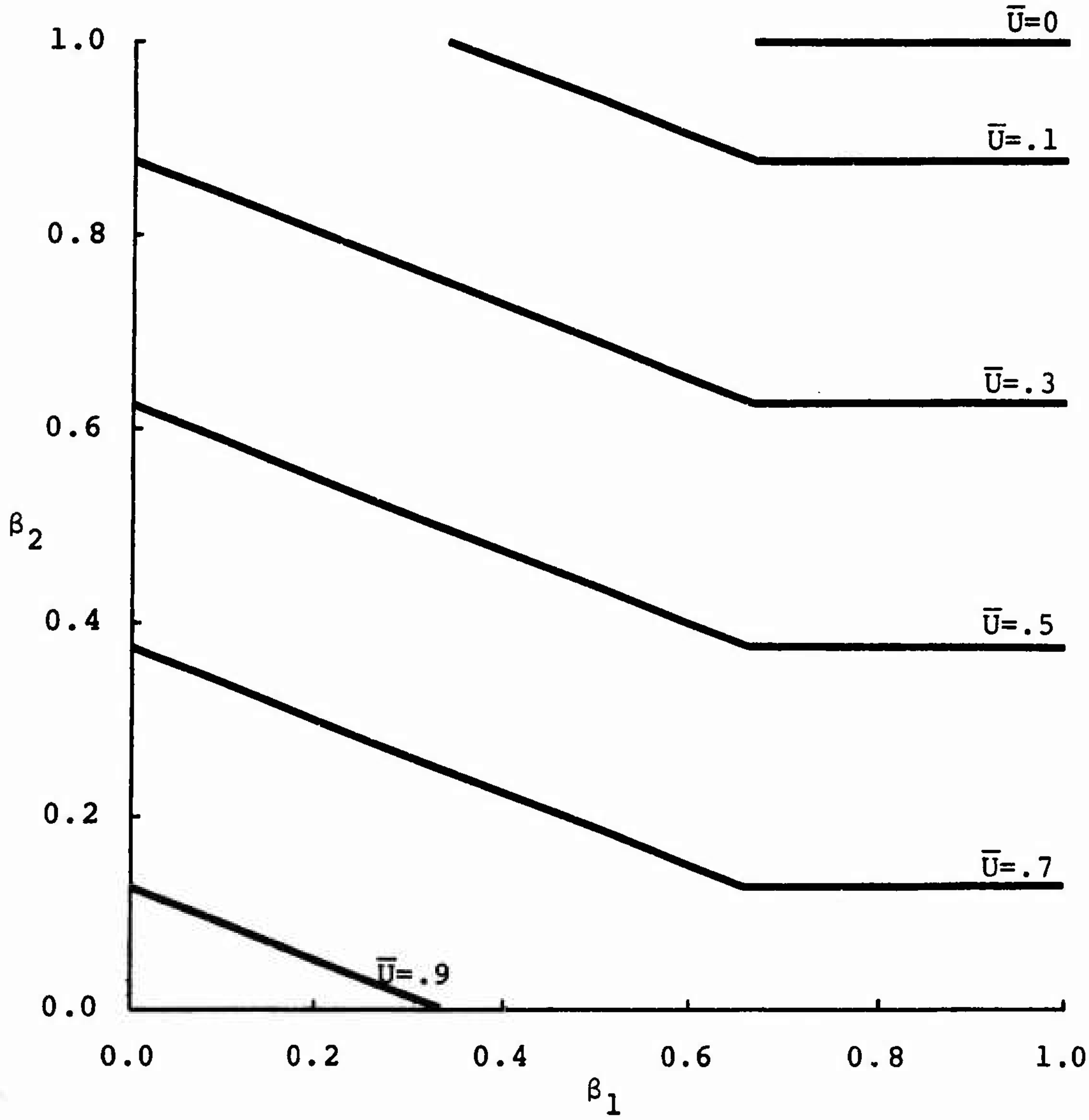

Figure 2.3c UR Model Iso-$\bar{U}$ Curves in (β_1, β_2) Plane

$p_1 = 0.2,\ d_1 = 1,\ f_1 = -.5,\ \bar{\beta}_1 = 2/3$

$p_2 = 0.8,\ d_2 = 1,\ f_2 = -.5,\ \bar{\beta}_2 = 2/3$

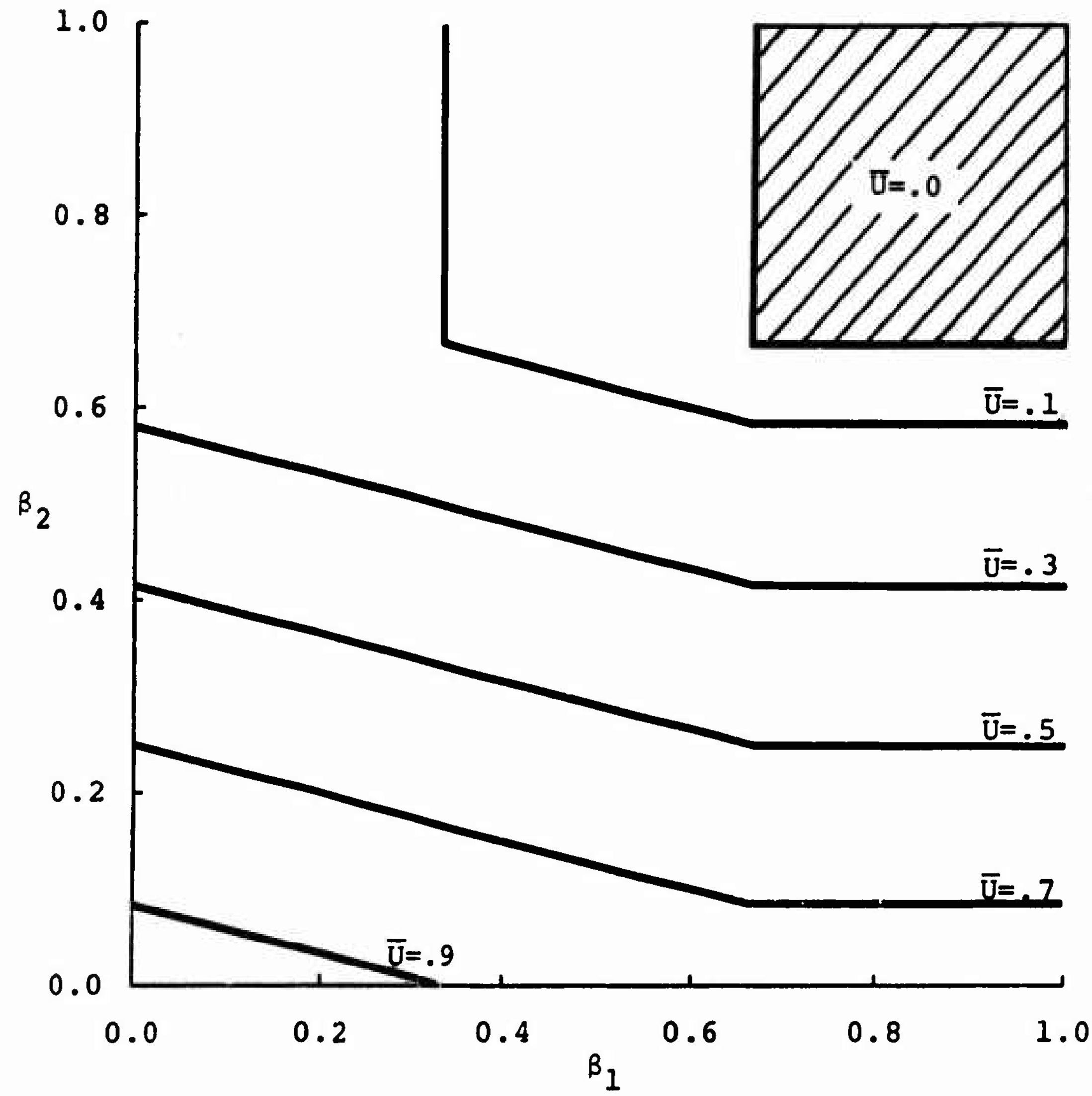

Figure 2.3d UR Model Iso-$\bar{U}$ Curves in (β_1, β_2) Plane

obtain the following relation for the portions of the iso-$\overline{U}$ curves having B_1 and B_2 positive:

$$p_1(d_1-f_1)\beta_1 + p_2(d_2-f_2)\beta_2 = p_1d_1 + p_2d_2-\overline{U}.$$

The iso-$\overline{U}$ curves are horizontal for $\beta_1 > \bar{\beta}_1$ and vertical for $\beta_2 > \bar{\beta}_2$ since the optimal search allocates no search time to box i if $\beta_i > \bar{\beta}_i$.

Consider the next sensitivity of $\overline{U}$ to the expected utility parameters, d_i and f_i. As long as $\beta_i < \bar{\beta}_i$

$$\frac{\partial \overline{U}}{\partial d_i} = \frac{\partial B_i}{\partial d_i} = p_i(1-\beta_i) \qquad (11)$$

and

$$\frac{\partial \overline{U}}{\partial f_i} = \frac{\partial B_i}{\partial f_i} = p_i\beta_i. \qquad (12)$$

To illustrate the interaction of these expected utility parameters with the false detection parameters, β_i, assume that $\beta_1 = \beta_2 = \beta$, $d_1 = d_2 = d$, and $f_1 = f_2 = f$. Then as long as B_1, $B_2 > 0$

$$\overline{U} = (1-\beta)d + \beta f. \qquad (13)$$

If f is fixed, this equation can be used to describe the tradeoff between β and d. If d is fixed it describes the

tradeoff between β and f. Figures 2.4a and 2.4b depict these tradeoff curves for $\bar{U} = .5$. These figures exhibit decreasing marginal returns for the β vs d tradeoff and increasing marginal returns for the β vs f tradeoff That is, the increase in d needed to compensate for a given increment in β is an increasing function of β while the corresponding decrease in f needed to compensate for an increment in β is a decreasing function of β.

2.2.3 *Discussion of UR Model Implications*

The introduction of the possibility of unfavorable response process outcomes shifts the optimal allocation from the Koopman model results, but the basic linear increasing character of the optimal allocation pattern remains. The effect of the unfavorable response outcomes is to multiply the target location probabilities, p_i, by correction factors. If the expected utility associated with searching in box i is negative ($B_i < 0$), the optimal search allocation avoids searching box i. If the Koopman search allocation is used for a problem with $B_i < 0$, the expected response results may be disasterous as in the case of Figure 2.2. However, if $B_1, B_2 > 0$, such disasterous results cannot be attained by any allocation. As long as $B_1, B_2 > 0$ the Koopman allocation yields nearly optimal expected utility values for amounts of available search time which are large compared to the expected

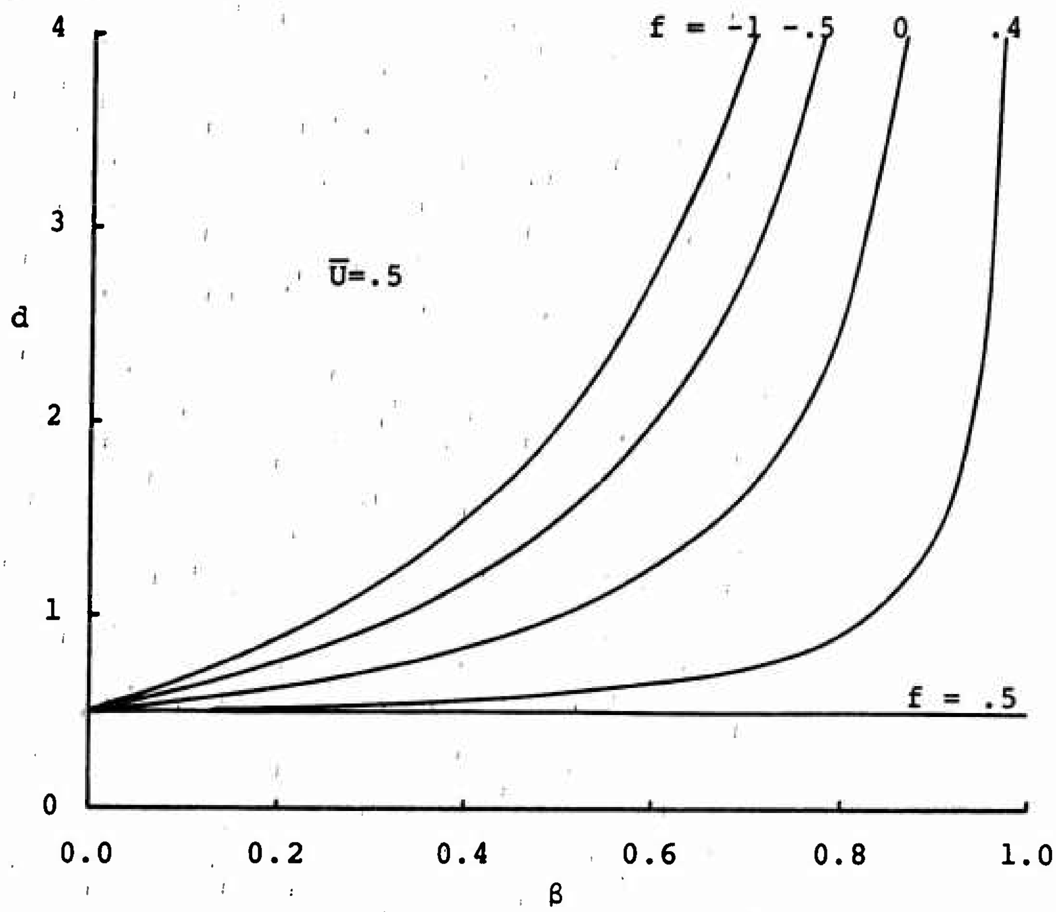

Figure 2.4a UR Model Iso-$\overline{U}$ Curves in (β, d) Plane

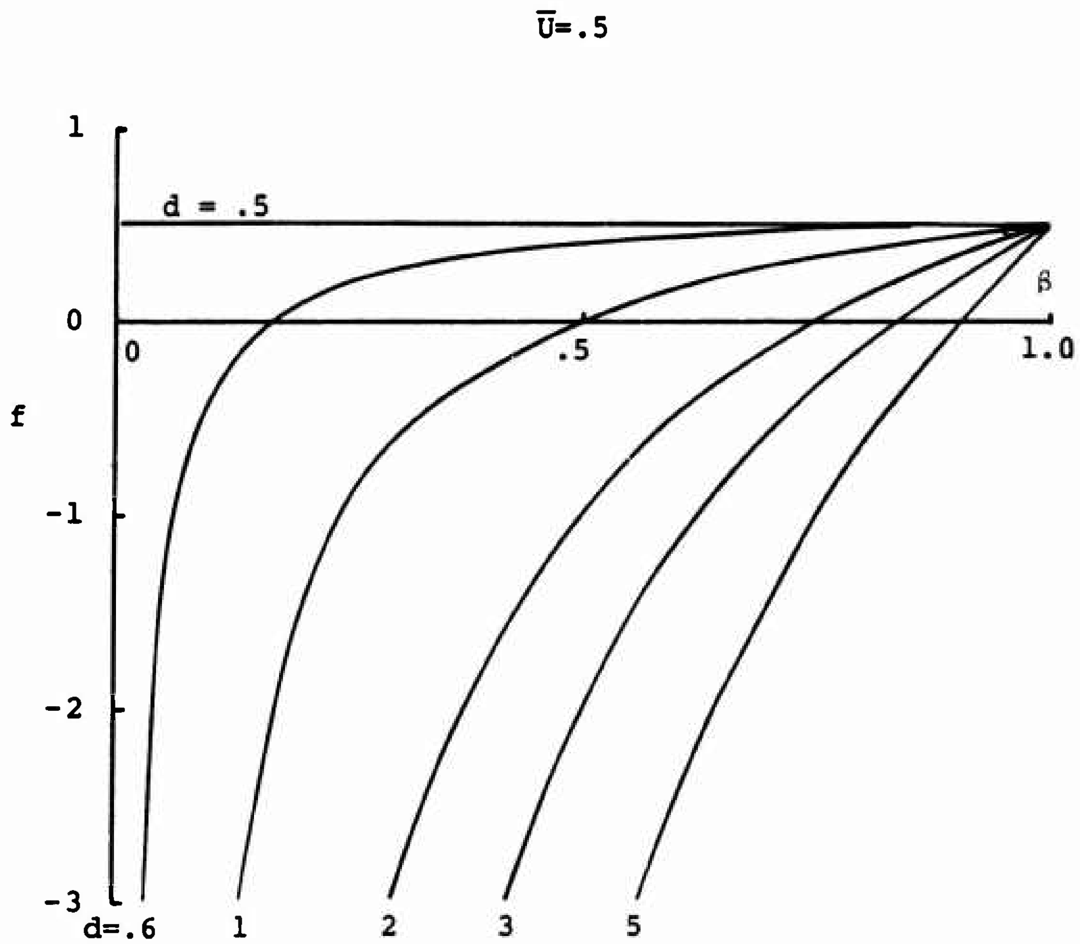

Figure 2.4b UR Model Iso-$\overline{U}$ Curves in (f, β) Plane

time to contact the target.

There exist cases for which the Koopman allocations are optimal for the UR allocation problem for any amount of available search time. If $(1-\beta_1)d_1 + \beta_1 f_1 = (1-\beta_2)d_2 + \beta_2 f_2$, this is the case. That is, the Koopman allocation is optimal if the conditional expected response utility, given that the target is contacted in box i, is the same for both boxes. Obviously, if the false detection and response process parameters for the two boxes are identical, this relation holds. This relation also holds if the expected utility associated with all possible contacts which result in searcher-target response interactions are the same ($d_1 = d_2 = f_1 = f_2$). For this case the optimal allocation and the optimal objective function are independent of the false detection parameters. For this situation the searcher is motivated to increase his contact rates, k_i, as high as is practical without being very concerned about keeping his false contact rates, β_i, low.

There exists a multidimensional continuum of parameter sets between the extreme cases in which the Koopman allocation is either optimal or disastrous. The combination of moderate parameter values which approach the disasterous result of Figure 2.2 represent cases involving high potential risk (β_i high, f_i negative) and only a modest potential gain (d_i small). On the other hand, ...

combinations of moderate parameter values lead to situations for which the optimal trajectory is nearly the same as the Koopman search plan. Thus, for large amounts of available search time, the inclusion of the false detection and response process phenomena in the model results in a large improvement in expected utility over that achievable using the Koopman model only in situations for which the Koopman model neglects obviously basic problem elements. That is, the main contribution of the more detailed UR model is in analyzing problems in which the available search time is quite limited.

If the false detection parameter, β_i, is sufficiently large that searching box i is harmful rather than helpful, the optimal search allocates no search time to box i. Therefore, the model results are completely insensitive to β_i if $\beta_i > \overline{\beta}_i$. But, if searching in box i is desirable ($\beta_i < \overline{\beta}_i$), the limiting optimal expected utility, $\overline{U}$, for large amounts of search time is a linearly decreasing function of β_i. Therefore, *if* the β_i values are adjustable with some tradeoff constraint relating achievable pairs of β_1 and β_2, equation 10 and the tradeoff constraint can be used to select the (β_1,β_2) pair which maximize $\overline{U}$. Note that (10) expresses the marginal value of decreasing β_i as a function of the prior probability distribution and response process parameters of the problem. Thus, if the

choice of the (β_1, β_2) pair must be made before the problem parameters are known, one needs to know the probability distribution of problem parameters as well to maximize the expected utility attainable by selecting (β_1, β_2).

The d vs β and the f vs β iso-$\bar{U}$ tradeoff relations (Figures 2.4a and 2.4b) provide a rational basis for allocating resources to obtain either search equipment quality (low β_i) or response system effectiveness (high d_i and/or f_i). Equation 13 can be used with the relevant description of the attainable combinations of β, d, and f to define optimal search-response systems. That is, assume that the set of efficient[1] combinations of β, d, and f parameters is expressed by $C(\beta, d, f) = 0$. Then, optimal choices of β, d, and f can be found by solving the mathematical programming problem

$$\text{Maximize } \bar{U} = (1-\beta)d + \beta f$$
$$\text{subject to } C(\beta, d, f) = 0.$$

Note that, unlike the corresponding β_1 vs β_2 selection problem, this problem does not involve the target location

[1]A set is efficient if one parameter cannot be made more desirable (lower for β, higher for d and f) except at the expense of at least one other parameter.

probability distribution[1]. Thus, the problem definition is more susceptible to quantification at the broader β vs d vs f level than the β_1 vs β_2 level.

For real systems we hypothesize that the cost associated with changing β alone rises very rapidly as approaches zero while the marginal costs associated with increasing d and f rise as d and f become large. If this is so, the increasing marginal benefit of reducing β exhibited by the f vs β iso-$\bar{U}$ curve is of no great importance[2]. If these cost effects are very pronounced for very small β and large d and f, the optimal search-response system has $0 < \beta < 1$.

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[1] One could consider each box separately integrating (10), (11) and (12) to obtain this same sort of optimization problem independent of the target location probability distribution.

[2] If costs were linear, this increasing marginal benefit characteristic would imply a "corner solution" -- $\beta = 0$ or 1.

CHAPTER 3

SEARCH MODELS WITH SCHEDULED RESPONSE (SR) PROCESSES

The previous chapter considered unscheduled response (UR) process search models. In this chapter we discuss similar models with scheduled response processes. For the UR model the response process only occurs following a contact event. In the SR models of this chapter we assume that the response process will occur following the search process regardless of the search outcome. After the search ends the decision maker guesses the target locatic and selects the response option which is conditionally optimal, given that his target location guess is correct. Then, the decision maker obtains a fixed reward from the ensuing response process if and only if the target location guess is correct.

The kinds of real problems which motivate the SR search models are illustrated by two scenarios:

Scenario 1 While playing unattended a small child is bitten by a dog. The dog escapes after being casually observed by several children. A significant percentage of stray dogs in the area are believed to have rabies. Thus, exposure to rabies is considered to be a threat to the small child's life. Since a rabid animal can be positively diagnosed before it is necessary to begin the risky

treatment indicated for a person who has been bitten by a rabid animal, a search for the dog is conducted before treatment is initiated. Of course, the child could be treated for exposure to rabies even if the dog is not found. But, the treatment itself could endanger the child's life if the child has not been exposed to rabies.

Scenario 2 In a war a defector presents convincing evidence that his homeland's army has brought a small number of nuclear weapons into the war zone for use at some designated time in the near future. But the defector does not know where the nuclear weapons are stored. The army which is threatened by this imminent nuclear attack diverts a substantial amount of its aerial reconnaissance effort to searching for information which would indicate which of his opponent's installations hides the nuclear weapons. If the location of the nuclear weapons is discovered, a maximum-effort conventional attack on this location has a substantial probability of destroying the nuclear weapons before they can be used.

The key elements of these problems are:

(i) The prime concern of the decision maker is in a response process which is not directly dependent on the search process.

(ii) False detections are an inevitable possibility for any reasonable search information processing system.

We consider three similar SR search optimization models which differ only in certain details. Each of these models concerns the use of Koopman "formula of random search" (exponential) type detectors to gather information regarding the location of a single target which can be in one of two locations or boxes. As in the UR models of Chapter 2 let

$$p_i = \Pr(\text{Target is in box } i),$$

$$e^{-k_i T_i} = \Pr(\text{No contact in box } i \mid \text{Target is in box } i),$$

$$T_i = \text{Amount of time allocated to searching in box } i.$$

For the UR model the contact event represents a decision to behave as if the target had been found. That is, when a contact occurred, an attempt was made to initiate the response process. For the SR models we will assume that contact events terminate the search process for the box in which the contact is made. But, the search process can continue in the other box. Only after the searches in both boxes are completed must the decision maker guess the target location. Thus, contacts can take place in both boxes.

False contacts in the SR models differ fundamentally from those of the UR model of Chapter 2. In this UR model

a false contact represented an initiation of the response process under unfavorable circumstances. Since the response process required the presence of the target, the UR model false contacts could only occur in the box containing the target. But, in the SR models false contacts represent detector data-processing "errors" which tend to cause the decision maker to guess the wrong target location. Thus, *although noisy signals can occur,* false *contacts* are impossible in the box containing the target. The false contacts of the SR models are associated exclusively with the box which does not contain the target. We assume that searching in the box which does not contain the target will produce an exponentially distributed random tine-to-contact, i.e.,

$$e^{-\beta_i k_i T_i} = \Pr(\text{No contact in box } i \mid \text{Target is not in box } i)$$

Thus, β_i is the ratio of the contact rates without and with the target present in box i. (We assume that $\beta_i < 1$.) Further, the times-to-contact for the two boxes are conditinally independent, given the target location.

After the search is completed the decision maker guesses the target location and selects the corresponding conditionally optimal response behavior pattern. If his guess is correct, he obtains an expected utility reward of d units. If his guess is incorrect, he obtains no reward.

The decision maker of this model makes two decisions: a search decision and a response decision. He first allocates the available searching time between two boxes. Then, after the search has been conducted, he selects his target location guess. Since the decision maker obtains utility only in the response process, the objective for the search allocation decision must be to maximize the resulting expected utility from the response process. The response (target location guess) decision is a simple decision problem involving risk. Given any possible search and search outcome, the decision maker can use Bayes' theorem to combine the prior target location probability distribution. Then, since the reward for correctly guessing the target location is the same for both boxes, it is optimal to guess that the target is in the box having the higher posterior probability of containing the target.

The three SR search models of this chapter differ in the assumptions made regarding the search outcome. The search outcome is described by the amounts of search time used in each box and the amounts of time allocated to these boxes. For a single reconnaissance plane taking aerial photographs continuously, the search time at which each picture was made is easily established. Thus, if a single picture provides the evidence on which the contact is based, the times-to-contact are available for use in making the target location guess decision. But, if the

contact represents an interpretation of the whole series of pictures with no single picture playing a dominant role in the interpretation, then the search times-to-contact have little meaning in terms of easily measured data. Therefore, it may be necessary to make the target location guess decision based only on the search allocation and which of the two boxes yielded contacts.

For aerial photo reconnaissance, the interpretation of the search data follows the search. In contrast, for a visual search by an observer in a light plane, the observer continuously interprets the data as the search is conducted. Thus, such a search system may be able to replan a search based on the preliminary search results as the search progresses. We analyze a version of our SR model with searching times-to-contact unavailable for making the target location guess and a version with adaptive search replanning as well as the basic version with searching times-to-contact available but no adaptive search replanning.

3.1 Limited Search Information (LSI) SR Model

First, let us analyze the decision maker's search allocation problem assuming that the only information derived from the search is the numbers of contacts which occur in each of the boxes. For i = 1,2 let

n_i = Number of contacts in box i, (Note that n_i = 0 or 1.),

(n_1,n_2) = Search outcome vector,

$P(m,n)$ = Pr(Search outcome is (m,n)),

$P(m,n \mid i)$ = Pr(Search outcome is (m,n) | Target is in box i), and

$p_i'(m,n)$ = Pr(Target is in box i | Search outcome is (m,n)).

The posterior target location probabilities can be computed using Bayes' theorem by

$$p_i'(m,n) = p_i \frac{P(m,n \mid i)}{P(m,n)} .$$

For the assumed search process

$$P(n_1,n_2 \mid i) = \left[e^{-k_iT_i}\right]^{1-n_i}\left[1-e^{-k_iT_i}\right]^{n_i} \cdot \left[e^{-\beta_jk_jT_j}\right]^{1-n_j}\left[1-e^{-\beta_jk_jT_j}\right]^{n_j} ,$$

where j = 3-i. So if the β_i are strictly positive[1],

[1]It will be shown later that the use of these posterior target location probability expressions leads to a correct statement of the problem even if β_1 or β_2, or both, are zero.

$$p_1'(0,0) = \frac{p_1}{P(0,0)} e^{-k_1T_1} e^{-\beta_2k_2T_2} , \tag{14a}$$

$$p_2'(0,0) = \frac{p_2}{P(0,0)} e^{-\beta_1k_1T_1} e^{-k_2T_2} , \tag{14b}$$

$$p_1'(1,0) = \frac{p_1}{P(1,0)} \left[1-e^{-k_1T_1}\right] e^{-\beta_2k_2T_2} , \tag{14c}$$

$$p_2'(1,0) = \frac{p_2}{P(1,0)} \left[1-e^{-\beta_1k_1T_1}\right] e^{-k_2T_2} , \tag{14d}$$

$$p_1'(0,1) = \frac{p_1}{P(0,1)} e^{-k_1T_1} \left[1-e^{-\beta_2k_2T_2}\right] , \tag{14e}$$

$$p_2'(0,1) = \frac{p_2}{P(0,1)} e^{-\beta_1k_1T_1} \left[1-e^{-k_2T_2}\right] , \tag{14f}$$

$$p_1'(1,1) = \frac{p_1}{P(1,1)} \left[1-e^{-k_1T_1}\right] \left[1-e^{-\beta_2k_2T_2}\right] , \tag{14g}$$

$$p_2'(1,1) = \frac{p_2}{P(1,1)} \left[1-e^{-\beta_1k_1T_1}\right] \left[1-e^{-k_2T_2}\right] . \tag{14h}$$

An optimal guess (response) decision is to guess that the target is in box i* such that[1]

$$p'_{i*}(m,n) = \max\{p'_1(m,n),\ p'_2(m,n)\} \quad , \qquad (15)$$

where the search outcome is (m,n).

The search objective is to maximize the unconditional probability of correctly guessing the target location. That is, the objective is

$$Z = \sum_{m,n} P(m,n)\ p'_{i*(m,n)}(m,n).$$

Substituting from (14) and (15)

$$\begin{aligned}
Z = &\left\{\max\ p_1 e^{-k_1T_1} e^{-\beta_2 k_2 T_2},\ p_2 e^{-\beta_1 k_1 T_1} e^{-k_2T_2}\right\} \\
&+ \max\left\{p_1\left[1-e^{-k_1T_1}\right]e^{-\beta_2k_2T_2},\ p_2\left[1-e^{-\beta_1k_1T_1}\right]e^{-k_2T_2}\right\} \\
&+ \max\left\{p_1 e^{-k_1T_1}\left[1-e^{-\beta_2k_2T_2}\right],\ p_2 e^{-\beta_1k_1T_1}\left[1-e^{-k_2T_2}\right]\right\} \\
&+ \max\left\{p_1\left[1-e^{-k_1T_1}\right]\left[1-e^{-\beta_2k_2T_2}\right]\right. , \qquad (16) \\
&\qquad \left. p_2\left[1-e^{-\beta_1k_1T_1}\right]\left[1-e^{-k_2T_2}\right]\right\} .
\end{aligned}$$

[1]The extension to consider separate rewards for each box follows directly with no complications.

The search allocation problem is to maximize Z given (16) subject to $T_1, T_2 \geq 0$, $T_1 + T_2 \leq T$.

Solution for $\beta_1 = \beta_2 = 0$

Let us denote the objective function for the special case $\beta_1 = \beta_2 = 0$ by Z^o. Then from (16) we obtain

$$Z^o = \max\left\{p_1 e^{-k_1 T_1}, \; p_2 e^{-k_2 T_2}\right\} + p_1\left[1-e^{-k_1 T_1}\right] + p_2\left[1-e^{-k_2 T_2}\right] \quad . \tag{17}$$

Note that the term of Z resulting from the (1,1) search outcome, which is impossible for $\beta_1 = \beta_2 = 0$, contributes nothing to Z^o. Therefore, the expression above obtained by substituting $\beta_1 = \beta_2 = 0$ in (16) is a correct expression of Z^o. Equation 17 may be rearranged as

$$Z^o = 1 - \min\left\{p_1 e^{-k_1 T_1}, \; p_2 e^{-k_2 T_2}\right\} . \tag{18}$$

From this expression the solution of the corresponding search allocation problem is obvious:

If $p_1 e^{-k_1 T} < p_2 e^{-k_2 T}$, $(T_1^*, T_2^*) = (T,0)$.

If $p_1 e^{-k_1 T} = p_2 e^{-k_2 T}$, $(T_1^*, T_2^*) = (T,0)$ or $(0,T)$.

If $p_1 e^{-k_1 T} > p_2 e^{-k_2 T}$, $(T_1^*, T_2^*) = (0,T)$

Important characteristics of the solution to this special case of the search allocation problem are:

1. The total amount of available search time, T, will always be allocated.
2. The optimal search is concentrated completely in one of the two boxes.
3. Considering T as a parameter, one of three possible cases occurs:
   a. If the boxes are identical ($p_1 = p_2$ and $k_1 = k_2$), it is optimal to search entirely in either box $(T_1^*, T_2^*) = (T,0)$ or $(0,T)$ for any $T > 0$.
   b. Let $j = 3-i$. If $p_i < p_j$, $k_i \geq k_j$ or $p_i = p_j$, $k_i > k_j$, $T_i^* = T$, $T_j^* = 0$ is optimal for any $T > 0$.
   c. If neither case a nor case b obtains, there exists i, $j = 3-i$, and

$$T^e = \frac{\ln \frac{p_i}{p_j}}{k_j - k_i} > 0$$

such tnat:

i. If $T < T^e$, $T_i^* = T$, $T_j^* = 0$.
ii. If $T = T^e$, $(T_1^*, T_2^*) = (T,0)$ or $(0,T)$.
iii. If $T > T^e$, $T_i^* = 0$, $T_j^* = T$.

This SR model optimal allocation is quite different from the Koopman allocation. For large T the Koopman allocations to the two boxes are approximately inversely proportional to their contact rates. But for the SR-LSI model with $\beta_1 = \beta_2 = 0$ (no false contacts) the optimal allocation is to concentrate the search in one box. While the Koopman allocation to each box is non-decreasing as a function of T, the SR-LSI optimal allocation may switch from $(T_1^*, T_2^*) = (T,0)$ to $(T_1^*, T_2^*) = (0,T)$.

Solution for β_1, $\beta_2 > 0$

Consider next the case in which β_1 and β_2 are positive. Equation 16 defines the objective function in a piecewise fashion. The objective function is continuous and differentiable except at the boundaries between the pieces. But the number of pieces comprising the objective function and the boundaries between pieces are only implicitly specified by (16) -- they depend on the parameters of the problem. Therefore, a direct solution of the problem based on marginal methods is complicated by non-differentiability problems. We shall

simplify the expression of the problem before solving it.

The source of the difficulties we wish to eliminate from the problem is that $p'_{i*}(m,n)$ is defined as the maximum of two functions. Corresponding to each of these maximization operations is the optimal conditional guess plan, given a particular search outcome. To remove the troublesome maximization operations we consider the set of allocation problems corresponding to each possible guess plan. Let S and R be the search outcome and response guess spaces (sets), i.e.,

$$S = \{(0,0), (0,1), (1,0), (1,1)\},$$
$$R = \{1,2\}$$

Each guess plan corresponds to a function mapping S into R. Let G be the set of all such functions, i.e.,

$$G = \{g \mid g: S \to R\}.$$

For any g in G, s in S let g(s) denote the image of s corresponding to the function g. The response guess plan corresponding to g is to guess that the target is in box g(s) whenever the search outcome is s. If response guess plan g is selected, the probability of

correctly guessing the target location is

$$Z^g = \sum_{s \text{ in } S} P(s)\, p'_{g(s)}(s). \qquad (19)$$

Consider the relationship between the set $\{Z^g \quad g \text{ is in } G\}$ and the objective function, Z. For any particular set of parameters, β_i, k_i, p_i, and any allocation, (T_1,T_2), there is at least one g in G such that $z^g(T_1,T_2) = Z(T_1,T_2)$. That is, some guess plan must be optimal for any particular allocation. Therefore, corresponding to the optimal solution, (T_1^*,T_2^*), there is an optimal guess plan, g*, such that $Z^{g^*}(T_1^*,T_2^*) = Z(T_1^*,T_2^*)$. Let $(T_1^{g^*},T_2^{g^*})$ be an optimal solution to the guess-plan constrained allocation problem with Z^{g^*} replacing Z as the objective function. Then, since the feasible regions for this related problem and the decision maker's allocation problem are identical,

$$Z^{g^*}(T_1^{g^*},T_2^{g^*}) \geq Z^{g^*}(T_1^*,T_2^*) = Z(T_1^*,T_2^*).$$

But, by the definitions of Z^{g^*} and Z

$$Z(T_1^{g^*},T_2^{g^*}) \geq Z^{g^*}(T_1^{g^*},T_2^{g^*}),$$

and by the definition of (T_1^*,T_2^*)

$$Z(T_1^{g^*},T_2^{g^*}) \leq Z(T_1^*,T_2^*).$$

Therefore, $Z(T_1^{g^*},T_2^{g^*}) = Z(T_1^*,T_2^*)$ and $(T_1^{g^*},T_2^{g^*})$ is optimal for the decision maker's problem. Clearly all solutions to the decision maker's problem can be obtained by computing all solutions to each related guess-plan constrained problem and selecting those solutions which yield $Z^g(T_1^g,T_2^g) = \max_{h \text{ in } G} \{Z^h(T_1^h,T_2^h)\}$, where (T_1^h,T_2^h) is any optimal solution to the related guess-plan constrained allocation problem corresponding to function h in G.

The set of possible guess plans, G, contains 16 members. We wish to avoid having to solve the 16 corresponding guess-plan constrained allocation problems in order to find the solutions to the decision maker's problem. Most of the possible guess plans can be eliminated as potential optimal guess-plans without solving the corresponding guess-plan constrained allocation problems. If (14) and (15) are substituted into (19) for all 16 possible guess plans one can use the relation[1]

$$\frac{1-e^{-x}}{1-e^{-\beta x}} > \frac{1}{\beta}\, e^{-(1-\beta)x} \qquad \text{for } 0 < \beta < 1,\ 0 < x$$

[1]Derived in Appendix B

to eliminate 10 guess plans as being nowhere optimal. Further, if we seek *an* optimal solution, rather than *all* optimal solutions, four of the remaining six guess plans can be eliminated from consideration as follows: Suppose that for every feasible (T_1,T_2) there exists a feasible (T_1',T_2') such that

$$Z^h(T_1',T_2') \geq Z^g(T_1,T_2).$$

Then, guess plan g is weakly dominated by guess plan h; hence, plan g can be eliminated from consideration.

The two remaining candidate optimal guess plans are:

I Guess that the target is in box 1 unless search outcome is (0,1).

II Guess that the target is in box 2 unless search outcome is (1,0).

The objective function corresponding to plan I above is

$$Z = p_1 - p_1 e^{-k_1 T_1}\left[1-e^{-\beta_2 k_2 T_2}\right] + p_2 e^{-\beta_1 k_1 T_1}\left[1-e^{-k_2 T_2}\right]. \qquad (20)$$

Since candidate optimal guess plan II is of the same form as plan I with the roles of the boxes interchanged, the objective function corresponding to plan II is of the same form as (20) with the subscripts interchanged. Consequently, we have two similar guess-plan constrained

search allocation problems which we need to solve to solve the decision maker's allocation problem.

An efficient numerical algorithm for solving these two "guess plan constrained searcn allocation problems" is described in Appendix B. This algorithm is based on the following (necessary) geometric conditions which are equivalent to the applicable Kuhn-Tucker conditions:

i. If an interior point of the feasible region is optimal, the gradient of the objective function must be the null vector at this solution point.

ii. If a boundary point of the feasible region is optimal, either the gradient of the objective function is the null vector or this gradient vector is an outward normal to a support line of the feasible region at the solution point.

The numerical algorithm uses the following results for the particular form of feasible region and objective function for guess plan I:

1. There is a unique point, (T_1^+, T_2^+), above the T_1 axis at which the gradient of the objective function is the null vector. The coordinates of this point may be expressed in terms of the root of a continuous monotomic function.

2. If the point (T_1^+, T_2^+) is feasible, the second order marginal conditions for a relative maximum hold and (T_1^+, T_2^+) solves the guess plan constrained search allocation problem.

3. If the point (T_1^+, T_2^+) is not feasible, the solution is on the boundary of the feasible region. There exists T_1^0 such that the gradient of the objective function is in the negative T_2 direction for all $(T_1, 0)$ with $T_1 < T_1^0$. If $T_1^0 \geq T$, all points $(T_1, 0)$ such that $0 \leq T_1 \leq T$ solve the guess-plan I search allocation problem and yield an optimal objective function value $Z^* = p_1$. If $T_1^0 < T$, there exists a unique boundary point, (T_1^*, T_2), with $T_2^* > 0$, which satisfies the marginal necessary condition for a boundary solution and which yields a value of the objective function which is greater than p_1. (T_1^*, T_2^*), then, is the unique solution to the guess-plan I search allocation problem. The coordinates of this solution point may be computed by finding the root of one of two continuous monotonic functions.

3.1.1 Comparison of SR-LSI and Koopman Model Results

Based on the solution algorithm developed in Appendix B for the two guess-plan constrained search allocation problems, a FORTRAN IV computer program was coded for the MTS IBM 360 to compute approximate numerical solutions to the LSI version of the decision maker's allocation problem.

Figures 3.1a to 3.1 d depict representative optimal allocations as functions of the amount of available search time, T. These allocation functions are represented by traces of the solution point, (T_1^*,T_2^*), which we call optimal trajectories. For any positive value of T, the optimal allocation is given by the point on the optimal trajectory having the maximum (T_1+T_2) value not exceeding T. These optimal trajectories consist of alternating segments of conditionally optimal trajectories corresponding to the two candidate optimal response guess plans used as the basis of the computational scheme. We use the terms "conditional trajectory I" and "conditional trajectory II" to refer to the conditionally optimal trajectories corresponding to guess plans I and II. These conditional trajectories are indicated by the designations I and II in Figure 3.1. Each of the conditional trajectories is continuous consisting of one or both of two segments: The first segment is a portion of one of the axes from the origin to some positive value;

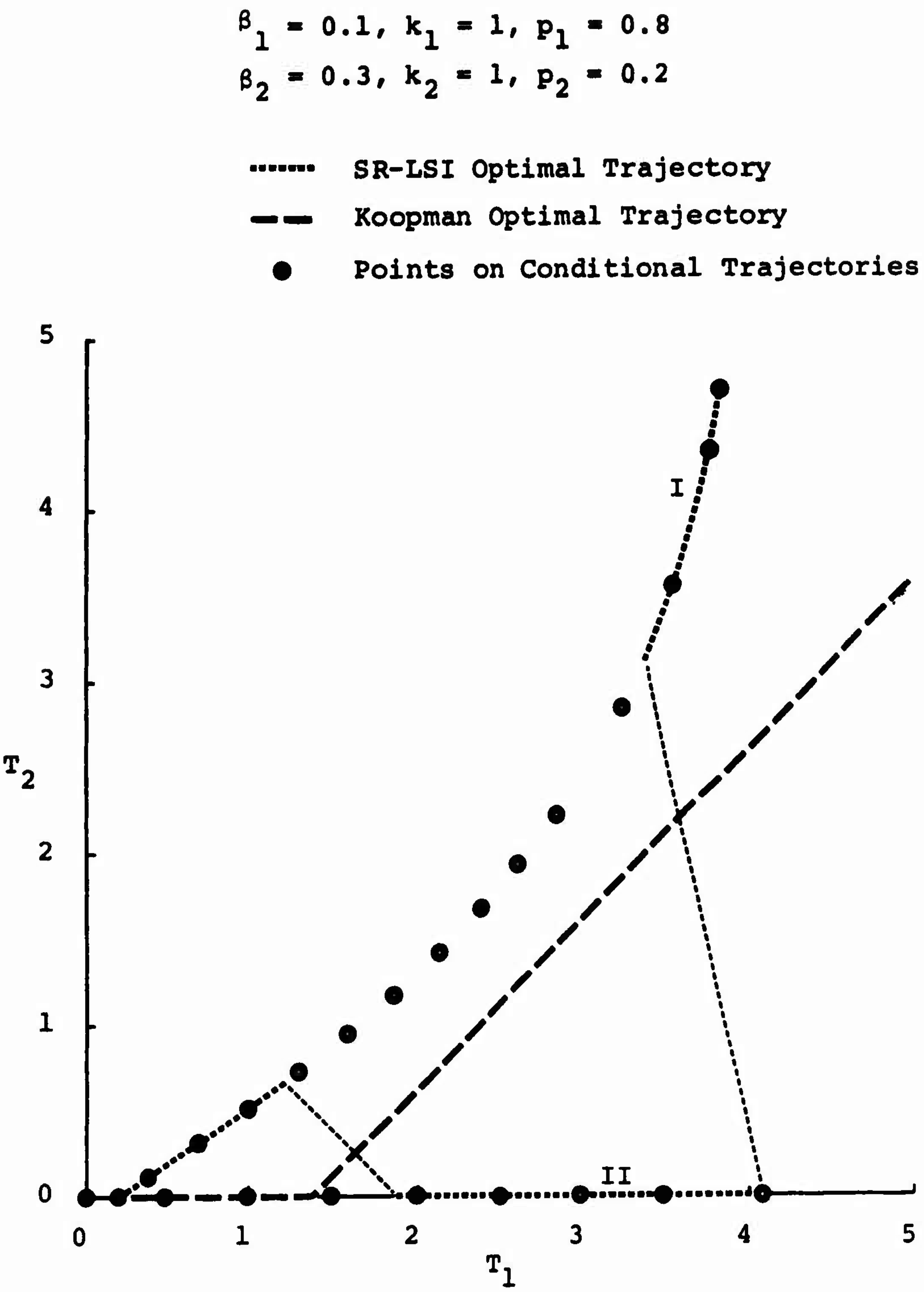

Figure 3.1a SR-LSI Model and Koopman Optimal Trajectories

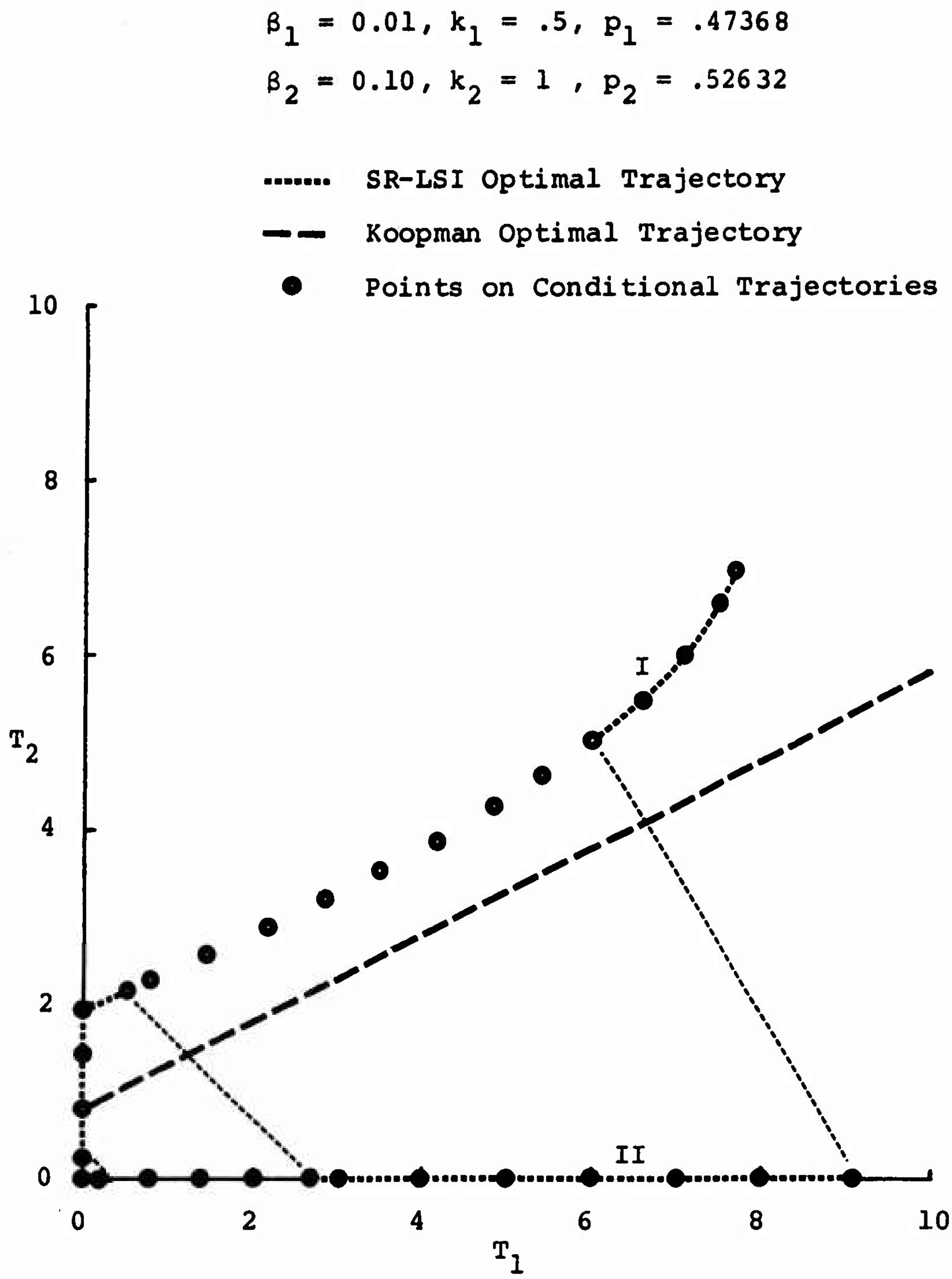

Figure 3.1b SR-LSI Model and Koopman Optimal Trajectories

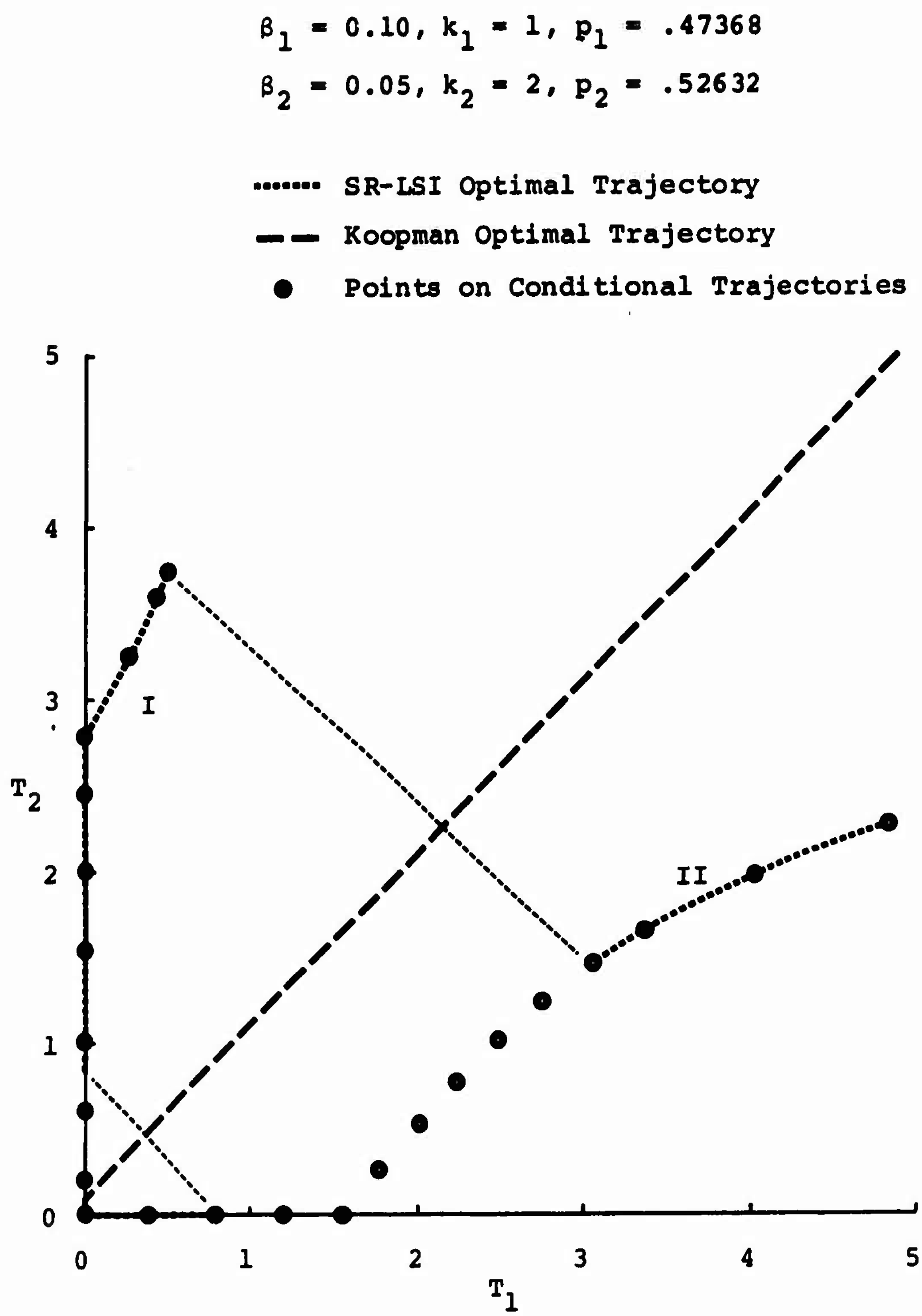

Figure 3.1c SR-LSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.1,\ k_1 = 1,\ p_1 = 0.5$

$\beta_2 = 0.5,\ k_2 = 1,\ p_2 = 0.5$

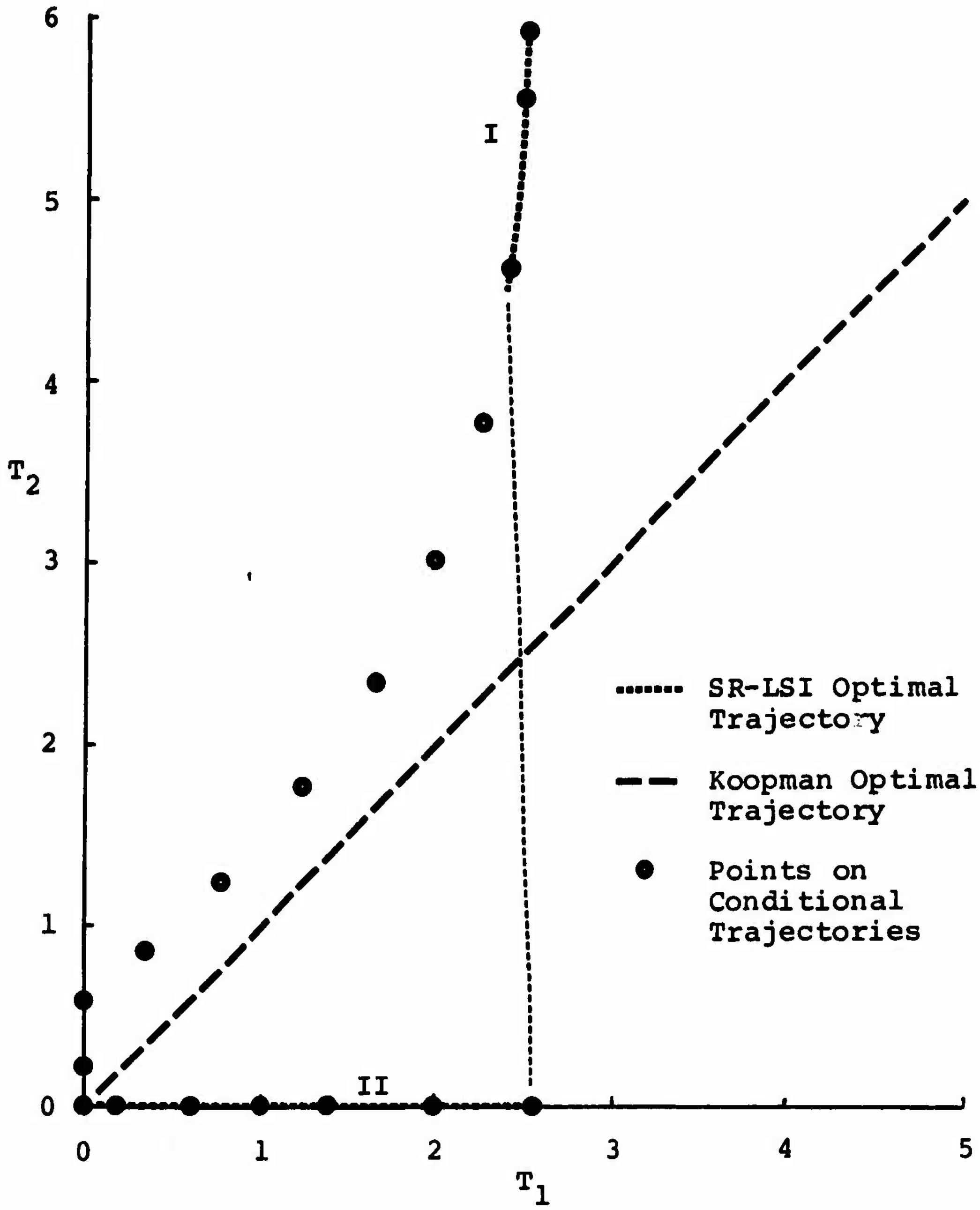

Figure 3.1d SR-LSI Model and Koopman Optimal Trajectories

the second segment is a bounded arc having continuous monotonic slope. Since the two conditional trajectories (and hence the optimal trajectory) are bounded, the amount of available search time allocated by the optimal search as T increases is bounded by an upper limit, $\bar{T}$.

Unless the false contact parameters, β_i, are both near one, $\bar{T}$ is large compared to the expected time to contact the target using the Koopman search plan. Further, for cases with moderate parameter values, the Koopman search plan trajectory for $0 < T < \bar{T}$ lies in the region bounded by the two conditional trajectories. Thus, the Koopman search allocations are usually in the same general region of the (T_1,T_2) plane as are the optimal allocations.

The optimal trajectory switches from one conditional trajectory to the other at points where the conditionally optimal objective function for one conditional trajectory overtakes that for the other conditional trajectory. The switch times corresponding to such optimal trajectory discontinuities can be determined numerically to any accuracy desired by computing the conditional trajectory points for values of T near these switch times. The computer program developed does not contain a search routine for isolating these switching times. The switching times used for plotting Figures 3.1a to 3.1d were obtained by graphic linear interpolation between the nearest points

for which computations were done. Computational experience has revealed that from zero to three switches between conditional trajectories can occur depending on the parameter values. Consider the interval between 0 and $\bar{T}$ (the amount of available searching time) as being divided roughly into four segments: small, moderate, moderately large and large. The following lexicographic rule correctly predicts the switching behavior observed for most of the cases examined with β_i, k_i and p_i parameters comparable for the two boxes:

i. For small T the optimal allocation is on the plan i conditional trajectory, where $p_i \geq .5$.

ii. For moderate T the optimal allocation is on the plan i conditional trajectory, where $k_i \leq k_j$, $j = 3-i$.

iii. For moderately large T the optimal allocation is on the plan i conditional trajectory, where $\beta_i \geq \beta_j$, $j = 3-i$.

iv. For large T the optimal allocation is on the plan i conditional trajectory, where $\beta_i \leq \beta_j$, $j = 3-i$.

In the event that the equality condition obtains for one of these conditions, the optimal allocation is indicated by one of the adajacent conditions. Condition i holds without exception. Conditions ii and iii hold for about 90% of the cases examined. Condition iv holds for about 60% of all the cases examined and for about 85% of the

cases examined with the prior probabilities the same order of magnitude ($.3 \leq p_1/p_2 \leq 3$).

Because optimal trajectories may switch between two conditional trajectories, the optimal allocations (T_1^* and T_2^*) may be non-increasing functions of T. Therefore, to determine an optimal search plan one needs to know the amount of search time which is available as well as the other parameter values. Hence, if T is not limited by some known bound, the optimal search plan may differ from both the Koopman and the UR model allocations.

For the same cases as Figures 3.1a to 3.1d, Figures 3.2a to 3.2d depict the SR-LSI expected utility values, Z_L, as functions of available search time, T, for both the optimal (SR-LSI) search allocation and the Koopman allocation. The ordinate scales in these plots are normalized so that the value at T = 0 is zero and the value corresponding to correctly guessing the target location with probability one is one. The difference between the SR-LSI curves and the Koopman curves represents the expected utility loss which would accompany the use of the Koopman model instead of the more complicated SR-LSI model. Also plotted in Figures 3.2a to 3.2d are the corresponding "perfect detector" expected utility functions -- those attainable with contact rates $(1-\beta_i)k_i$ with no false contacts. The difference between the optimal objective function curves and the perfect detector curves represents the loss attributable to the false contacts. Depending

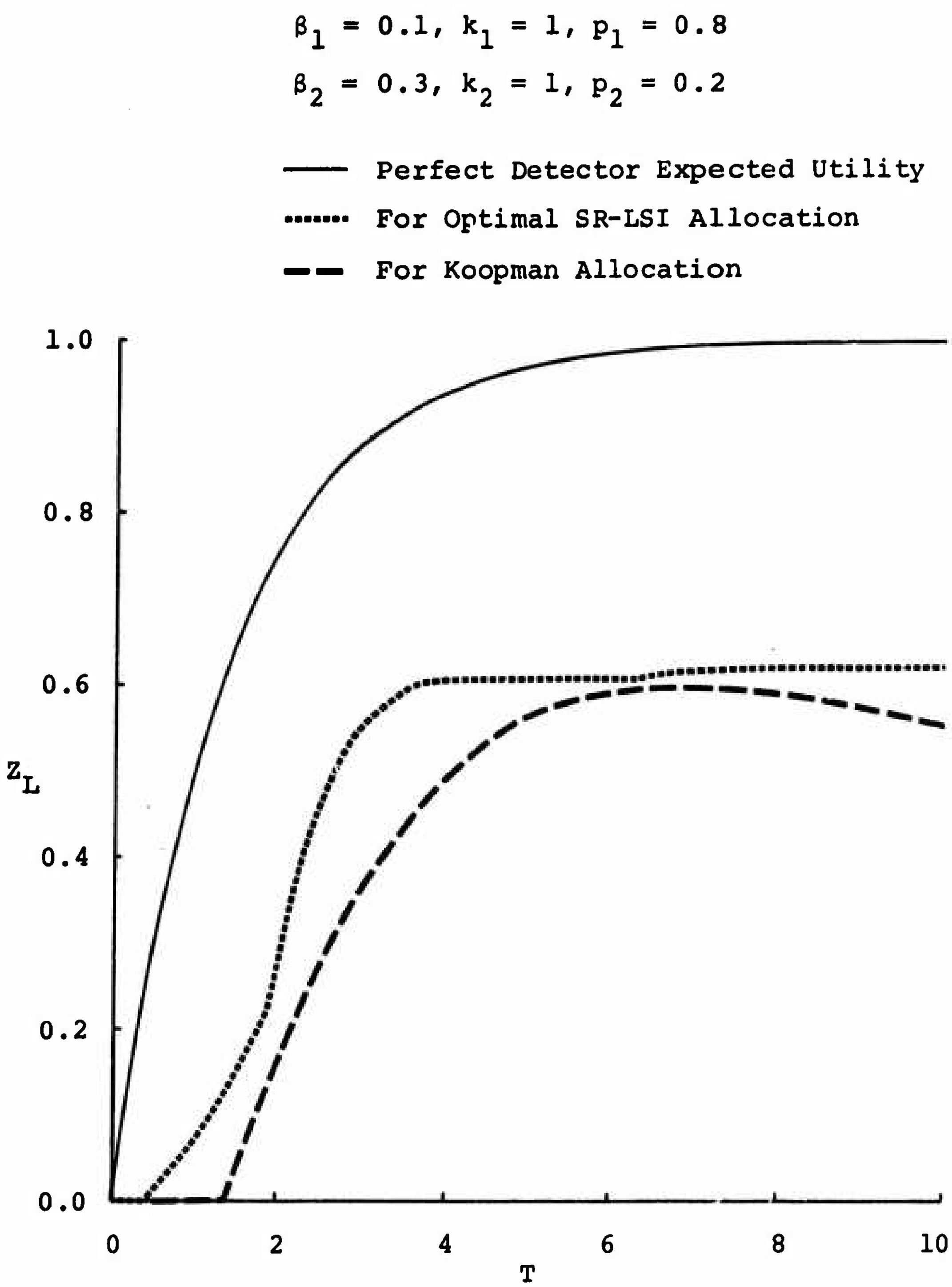

Figure 3.2a Normalized SR-LSI Model Expected Utility vs Available Search Time

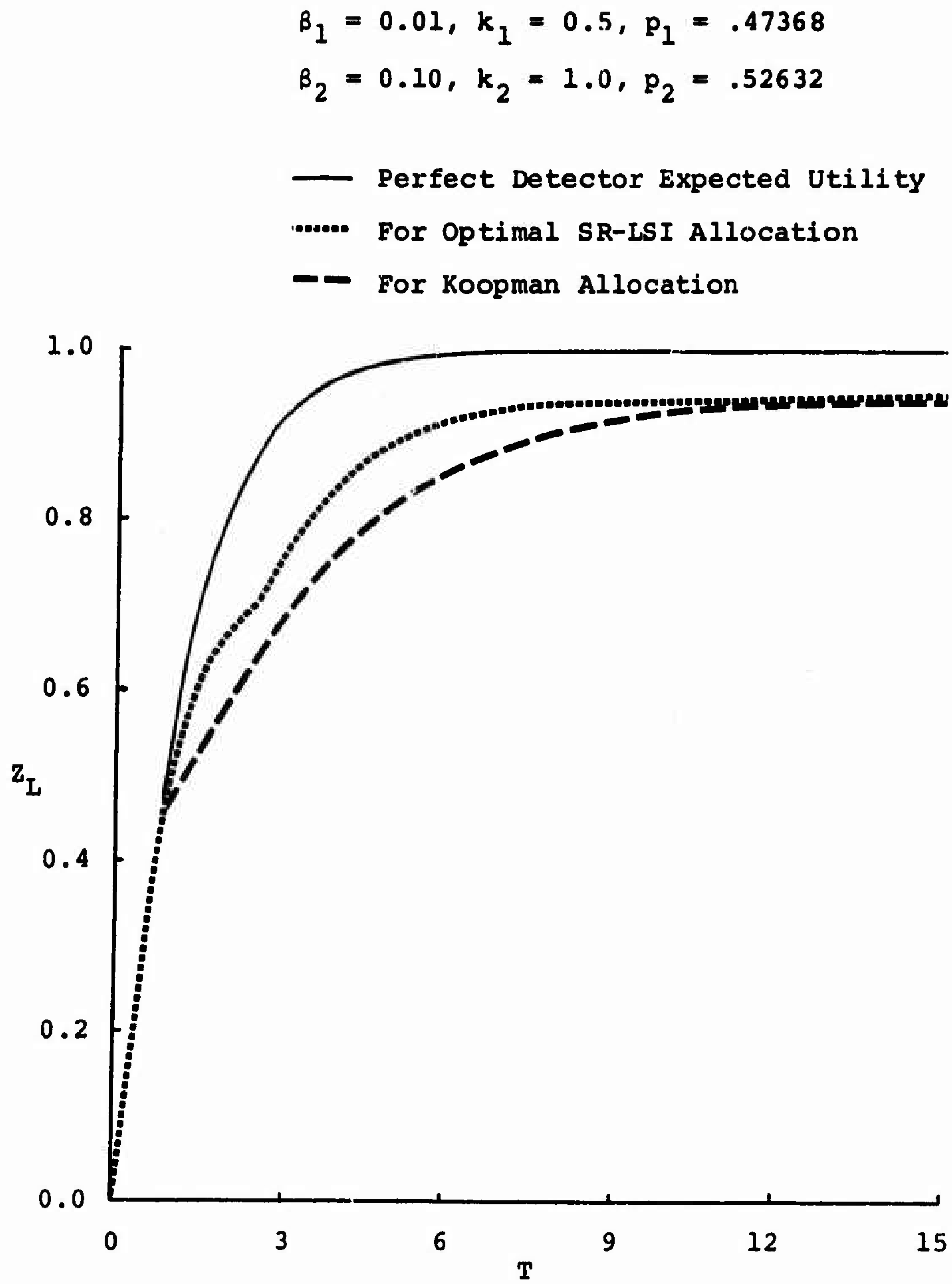

Figure 3.2b Normalized SR-LSI Model Expected Utility vs Available Search Time

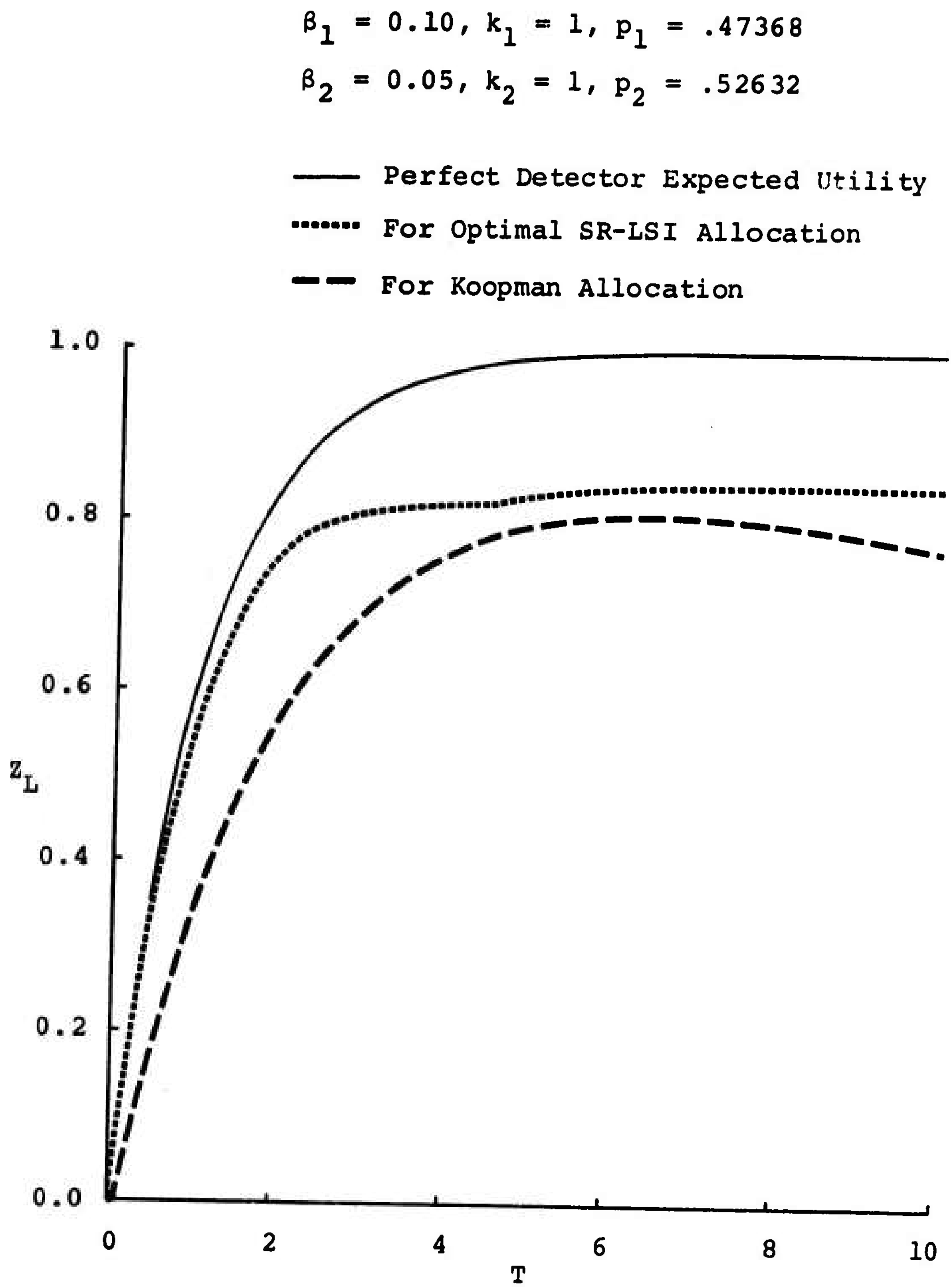

Figure 3.2c Normalized SR-LSI Model Expected Utility vs Available Search Time

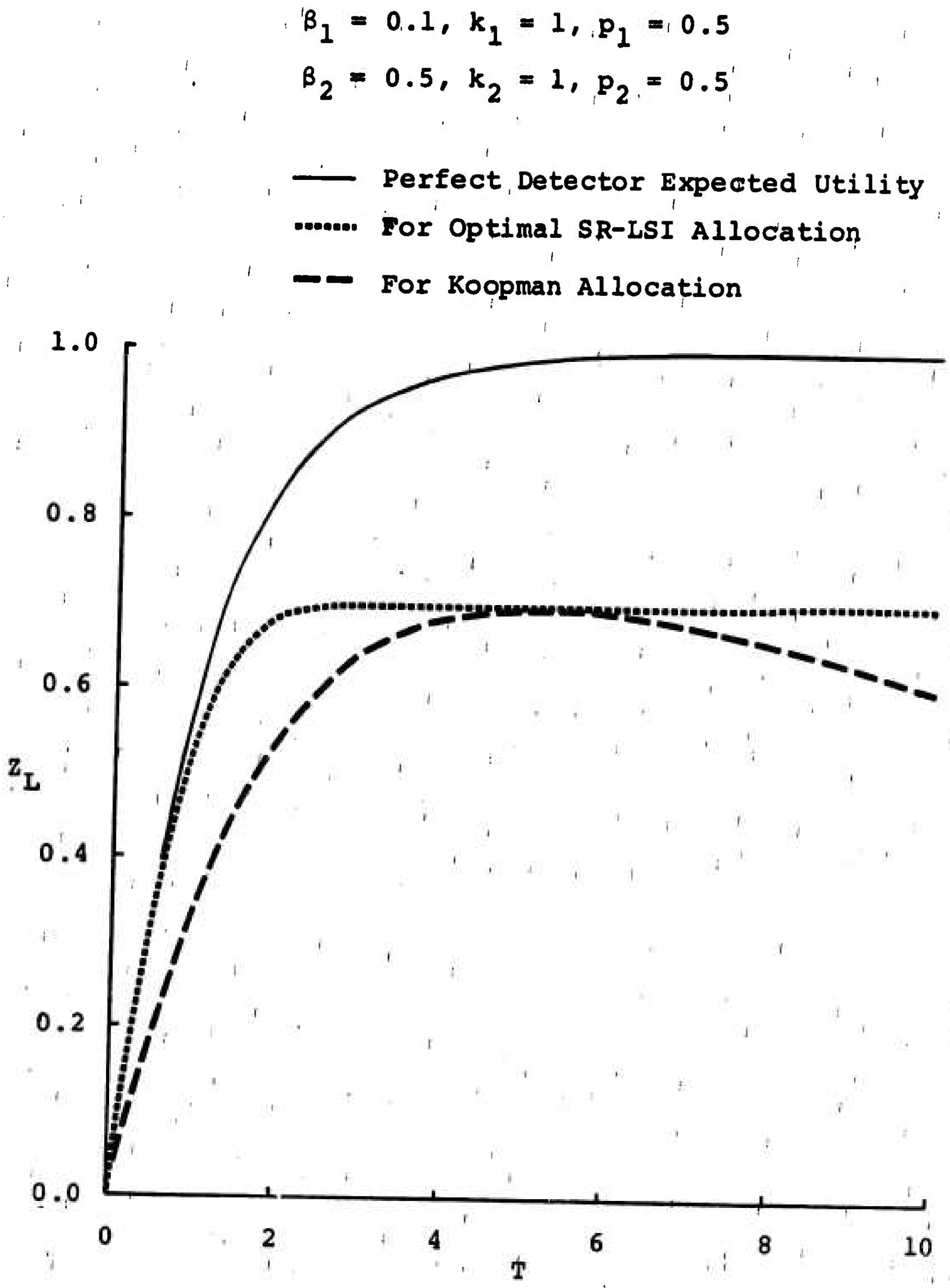

Figure 3.2d Normalized SR-LSI Model Expected Utility vs Available Search Time

on the model parameters, both the Koopman model expected utility loss and the expected utility loss die to false contacts vary from zero up to values of the same order of magnitude as the optimal expected utility. The loss due to false contacts is increasing in T for most cases. (In a few cases such as that shown in Figure 3.2b there exists an interval in which this loss decreases as T increases.) For large T the loss due to false contacts is of the same order of magnitude as the false contact parameters β_1 and β_2.

The loss from using the Koopman model is usually small for both small T and values of T approximately three to four times the expected search time to contact the target for the Koopman search plan. For practical systems the most important range of available search times is probably roughly from one-half to twice the expected time to contact. In this range, the loss from using the Koopman search plan is a complex function of the model parameters. For example, comparing Figures 3.2b and 3.2c we see that for T = 1 increasing β_1 and k_1 while decreasing β_2 results in an increase in the Koopman search normalized loss from almost nothing to about 20%. Also, comparing Figures 3.2c and 3.2d we see that increasing β_2 from 5% to 50% produced no noticable increase in the Koopman search loss. For small T, the Koopman loss is zero provided that the Kcopman and the SR-LSI optimal search allocations begin in the same box. This occurs if $p_i k_i > p_j k_j$,

where $i = 3-i$ and $p_i < p_j$. If k_i/k_j is sufficiently large that the interval for which the Koopman allocation is confined to box i is of the same order of magnitude as the Koopman search expected time to contact the target, the Koopman loss is relatively small for all values of T. On the other hand, if $p_i/k_i \cong p_j/k_j$ with $p_i \ncong p_j$, the Koopman search loss is relatively large for small and moderate values of T.

The optimal SR-LSI expected utility values as functions of available search time, T, increase continuously from zero to their maximum attainable values at finite times, $\bar{T}$. At the times for which the optimal trajectory switches from one conditional trajectory to the other, the slope of the optimal expected utility increases discontinuously. If the decision maker must pay a constant price, c_T, per unit of planned (available) search time, the optimal amount of search time can be obtained by finding the point or points where support lines of slope c_T touch the SR-LSI expected utility curves of Figure 3.2. Clearly, the corresponding optimal amounts of planned search time are non-increasing functions of c_T with discontinuities skipping each of the switch times. That is, there exist uneconomic available searching time intervals containing the switching times such that no time in these uneconomic intervals would be selected (optimal) for any value of c_T.

3.1.2 Sensitivity of SR-LSI Results

Behavior for Small T

The solution for small T can be computed[1] by replacing the objective functions by Taylor series expansions and solving the resulting mathematical programming problems. This analysis leads to the following results:

Let $i = 1,2$ and $j = 3-i$.

Case 1 $\beta_i \, p_j > p_i$ (Prior target location distribution dominates guess decision for small T.)

For small T, any feasible solution is optimal, and the optimal objective function value is $Z^* = p_j$.

Case 2 $\beta_i \, p_j = p_i$ (Borderline case between cases 1 and 3)

The unique optimal allocation for small T is

$$T_i^* = \frac{(\beta_i p_j - \beta_j p_i) k_j}{2(\beta_i p_j - \beta_j p_i) k_j + (p_i - \beta_i^2 p_j) k_i} \, T \, ,$$

$$T_j^* = T - T_i^* \quad .$$

And the corresponding optimal objective function value is

[1]See Appendix C for details.

$$Z^* \cong p_j + \left[(\beta_i p_j - \beta_j p_i) k_j T_j^* - (p_i - \beta_i^2 p_j) \frac{k_i}{2} T_i^* \right] k_i T_i^* .$$

Case 3 $\beta_i p_j < p_i$, $\beta_j p_i < p_j$, $p_i < p_j$ (Search outcome contributes to guess decision for all optimal allocations of positive T.)

The unique optimal allocation for small T is

$$T_i^* = T, \; T_j^* = 0,$$

and the corresponding optimal objective function value is

$$Z^* \cong p_j + (p_i - \beta_i p_j) k_i \, T. \tag{21}$$

Case 4 $p_i = p_j$ (Search outcome contributes to guess decision for any allocation.)

If $(1-\beta_j)k_j > (1-\beta_i)k_i$, the optimal allocation of small T is

$$T_i^* = 0, \; T_j^* = T,$$

and the corresponding optimal objective function value is

$$Z^* \cong .5 \left[1 + (1-\beta_j) k_j \; T \right]. \tag{22}$$

If $(1-\beta_1)k_1 = (1-\beta_2)k_2$ and $(1-\beta_i^2)k_i^2 < (1-\beta_j^2)k_j^2$,

the optimal allocation of small T is

$$T_i^* = T, \; T_j^* = 0,$$

and the corresponding optimal objective function value is

$$Z^* \cong .5\left[1 + (1-\beta_i)k_iT - \tfrac{1}{2}(1-\beta_i^2)k_i^2T^2\right] . \qquad (23)$$

If $(1-\beta_1)k_1 = (1-\beta_2)k_2$ and $(1-\beta_1^2)k_1^2 = (1-\beta_2^2)k_2^2$,

the parameters for the two boxes are identical. The optimal allocations of small T are

$$T_1^* = T, \; T_2^* = 0$$

and

$$T_1^* = 0, \; T_2^* = T,$$

and the corresponding optimal objective function value is given by (23) with i = 1 or i = 2.

These results for small T may be summarized as follows:

i. If no possible search outcome for any feasible search provides sufficiently strong evidence for changing the pre-search optimal guess (Case 1), any feasible search is optimal, but no possible search produces a positive increment in expected utility.

ii. If searching in only one of the boxes can produce sufficiently strong evidence to change the pre-search optimal guess (Cases 3 and 4), the optimal allocation of small T is confined to one box and the corresponding optimal objective

function value is approximately linearly increasing in T.

iii. For the intermediate condition between (i) and (ii) above (Case 2), the optimal allocation of small T is to allocate constant fractions of T (independent of T) to each of the two boxes. The corresponding optimal objective function value is quadratic in T with zero deriative at T = 0.

From (21), (22) and (23) it is clear that if (ii) obtains, the marginal contribution of small search effort with the β_i and k_i parameters fixed is maximized over all target location probability distributions by $p_1 = p_2$. In contrast, the absolute magnitude of the optimal objective function is maximized by $p_1 = 0$ or 1. Thus, if the prior target location probability distribution contains little information, the search can contribute significantly by collecting target information. But, the state of prior target location knowledge which leads to the highest expected utility is that of perfect information which renders the search useless. Little prior information implies relatively high values associated with searching but relatively low attainable expected utility values.

Behavior for large T

If $\beta_i = 0$, a contact in box i is known with certainty to be a detection. Therefore, as the search time grows without bound the probability of finding or infering

the target location approaches one. In contrast, if both β_1 and β_2 are positive, the true target location can never be known with certainty. Thus, the possibility of false contacts imposes a limit on the maximum attainable value of the expected utility even if the available amount of searching time is unconstrained. The sensitivity of this reduction in objective function value for large T to marginal changes in β_1 and β_2 is examined below.

A computer program was developed to compute the coordinates of points in the (β_1,β_2) plane along curves having constant (normalized) values of the SR-LSI optimal expected utility for large T, $\overline{Z}_L$. Figures 3.3a to 3.3c show such iso-$\overline{Z}_L$ curves for different values of the ratio p_1/p_2, the only relevant parameter in determining these curves.

These iso-$\overline{Z}_L$ curves exhibit the familiar "diminishing marginal utility" property. The extent of this effect, which is indicated by the rather sharp L-shaped character of these curves, is surprising. Since searching in a box having $\beta_i = 1$ is useless, we associate each end point of these iso-$\overline{Z}_L$ curves with a search concentrated completely in one of the boxes. The strongly diminishing marginal returns property of the iso-$\overline{Z}_L$ curves comes from the effectiveness of one box searches. That is, for large T the better of the two one-box searches usually achieves a value

$$\frac{P_1}{P_2} = 1$$

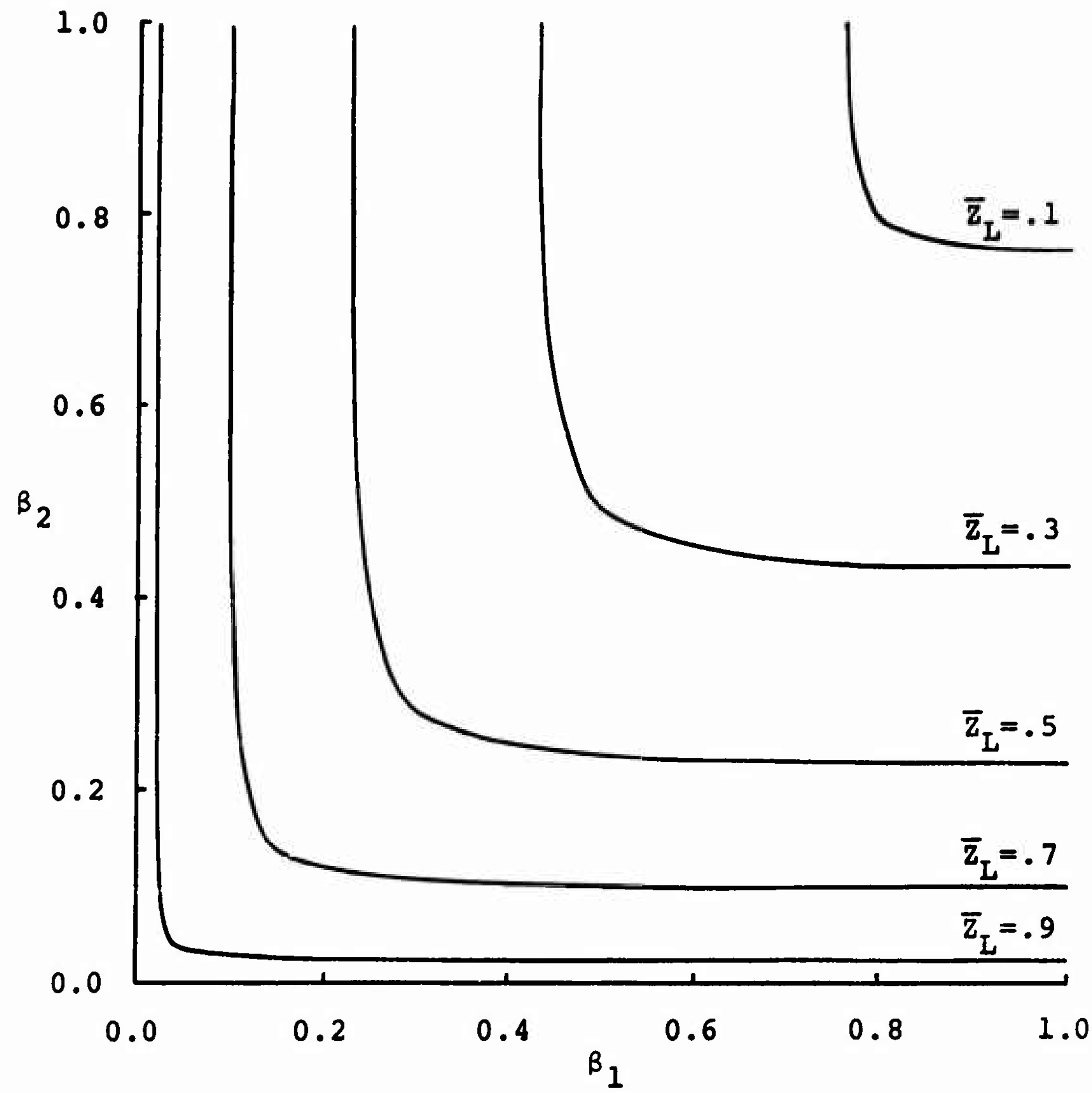

Figure 3.3a SR-LSI Model Iso-$\overline{Z}_L$ Curves in (β_1, β_2) Plane

$$\frac{p_1}{p_2} = 1.1$$

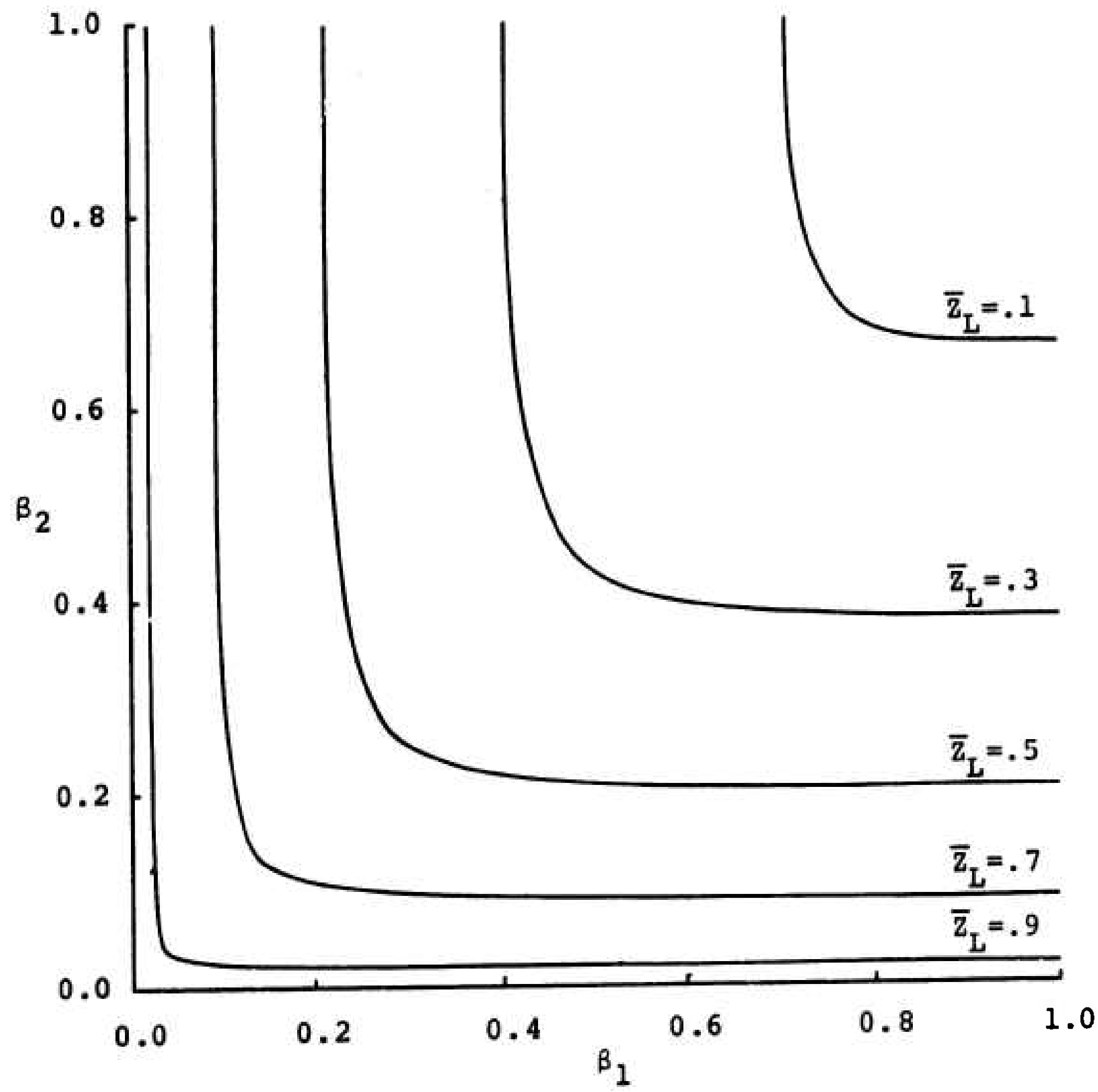

Figure 3.3b SR-LSI Model Iso-$\overline{Z}_L$ Curves in (β_1, β_2) Plane

$$\frac{P_1}{P_2} = 1.4$$

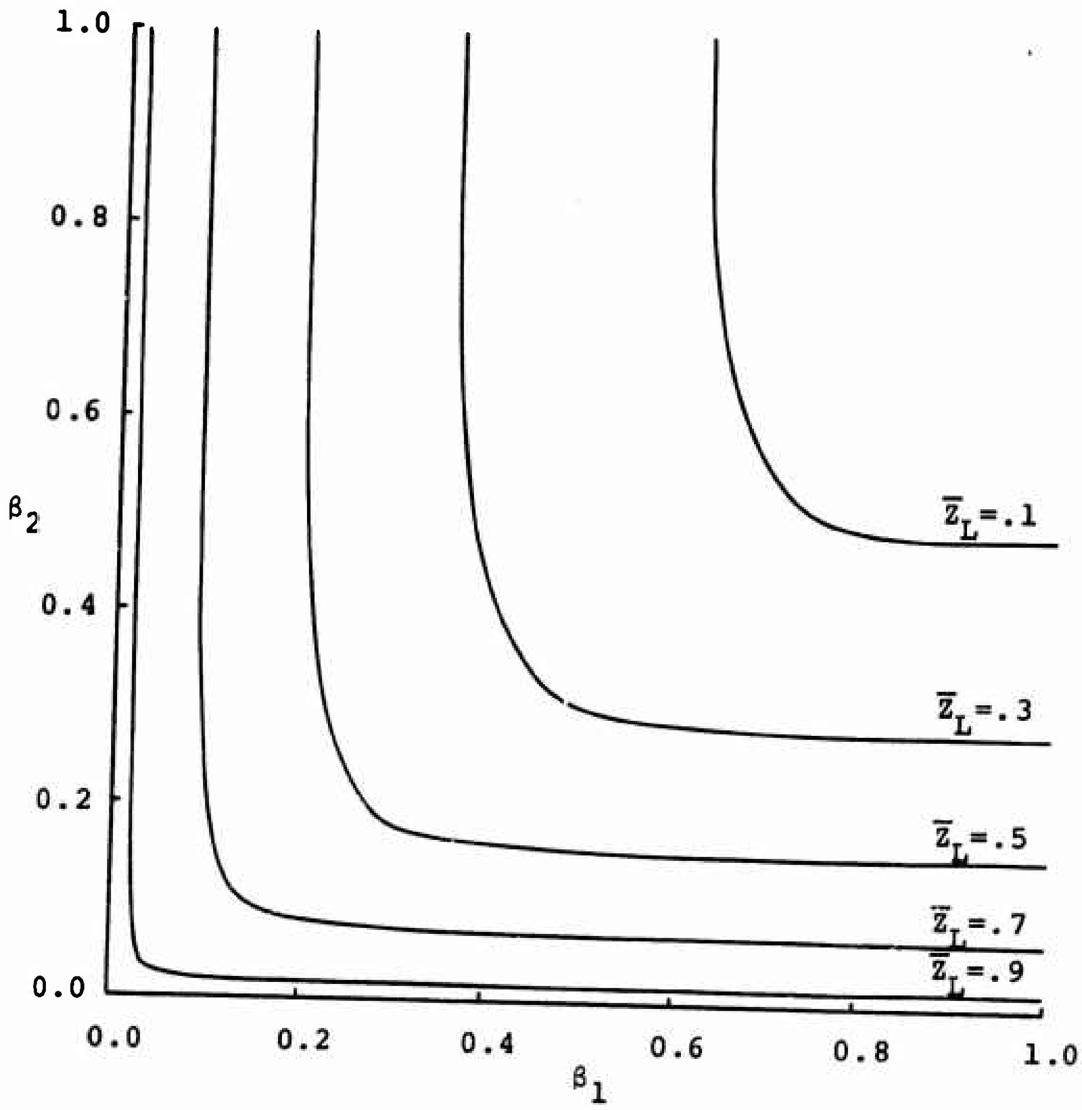

Figure 3.3c SR-LSI Model Iso-$\overline{Z}_L$ Curves in (β_1, β_2) Plane

of expected utility which is close to the optimal value. This phenomenon is particularly apparent when one of the false contact parameters, β_i, is very small.

A comparison of the sets of iso-$\overline{Z}_L$ curves reveals a dramatic shift toward the axes as the ratio p_1/p_2 increases from 1. The shift toward the β_1 axis is more extensive than toward the β_2 axis. Figure 3.4a illustrates this shift for the 30% iso-$\overline{Z}_L$ curves. Figures 3.4b and 3.4c show the shift in end points of these iso-$\overline{Z}_L$ curves as functions of the ratio p_1/p_2.

This phenomenon can be explained as follows: Lower β_i values correspond to more reliable or better quality search systems. For any set of search system parameters, the (normalized) $\overline{Z}_L$ value is maximized by the prior probability distribution $p_1 = p_2$. That is, any search system can attain a higher normalized expected utility value for $p_1 = p_2$ than for any other distribution. The dramatic shift in iso-$\overline{Z}_L$ curves expresses a marked drop in the attainable normalized expected utility as the probability distribution departs from $p_1 = p_2$. Or, as the probability distribution departs from $p_1 = p_2$, the search system quality needed to attain any given normalized iso-$\overline{Z}_L$ level rises dramatically. That is, searching is more valuable if the state of pre-search target location knowledge is that of complete ignorance.

The cause of this dramatic change is embedded in the

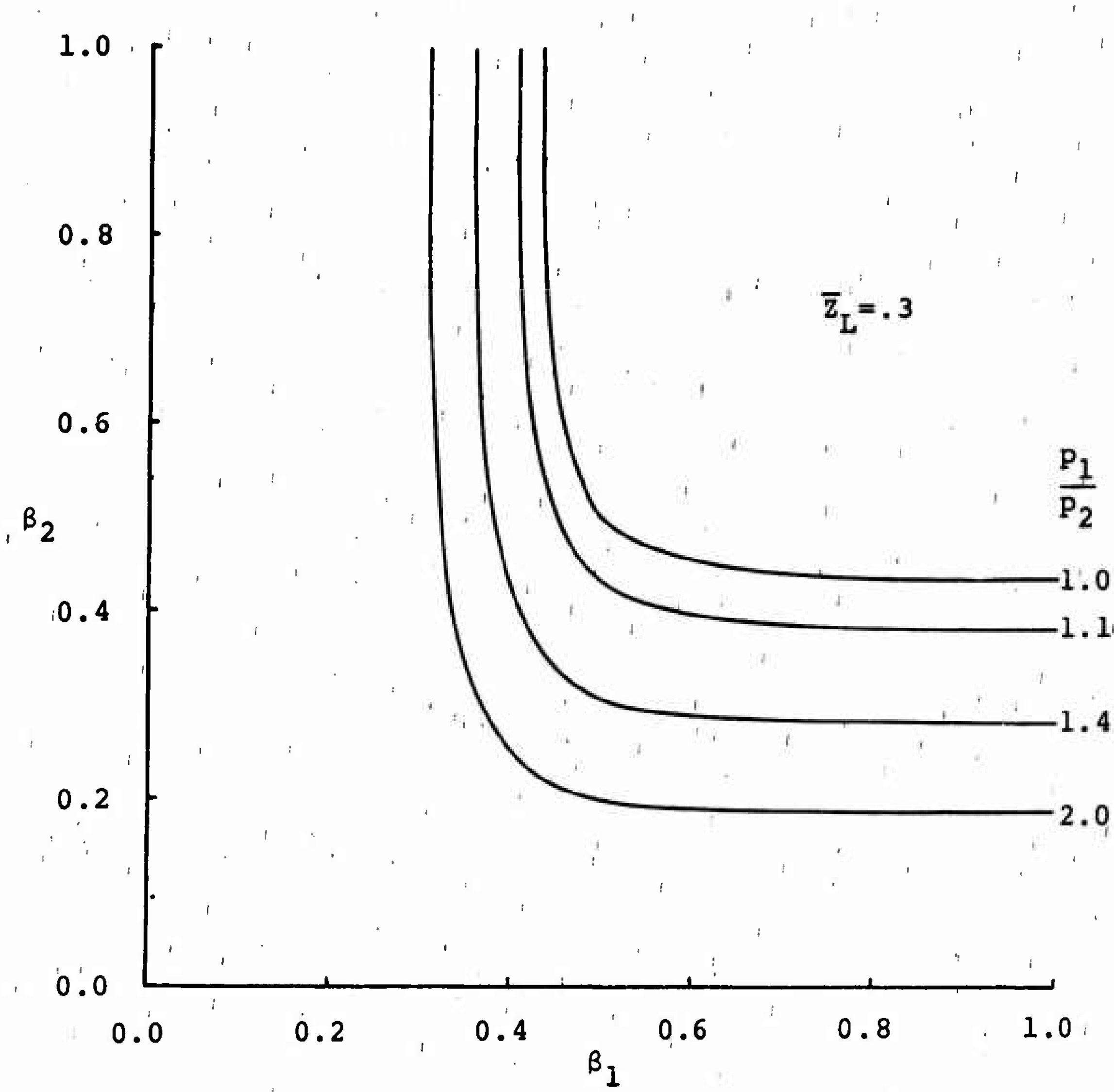

Figure 3.4a Dependence of SR-LSI Model 30% Iso-$\overline{Z}_L$ Curves on p_1/p_2

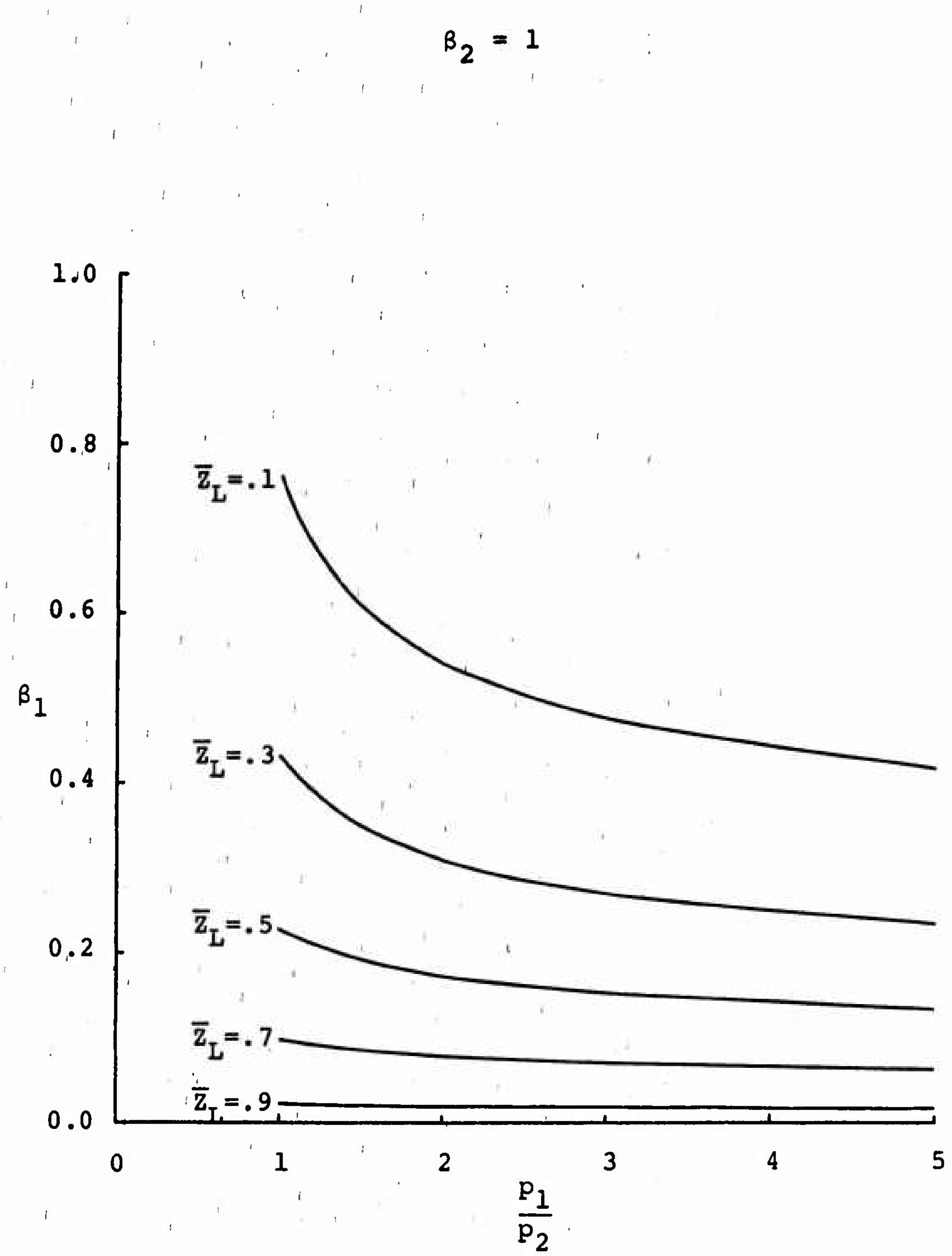

Figure 3.4b Dependence of SR-LSI Model $\beta_2 = 1$ Iso-$\bar{Z}_L$ Curve Endpoints on p_1/p_2

$\beta_1 = 1$

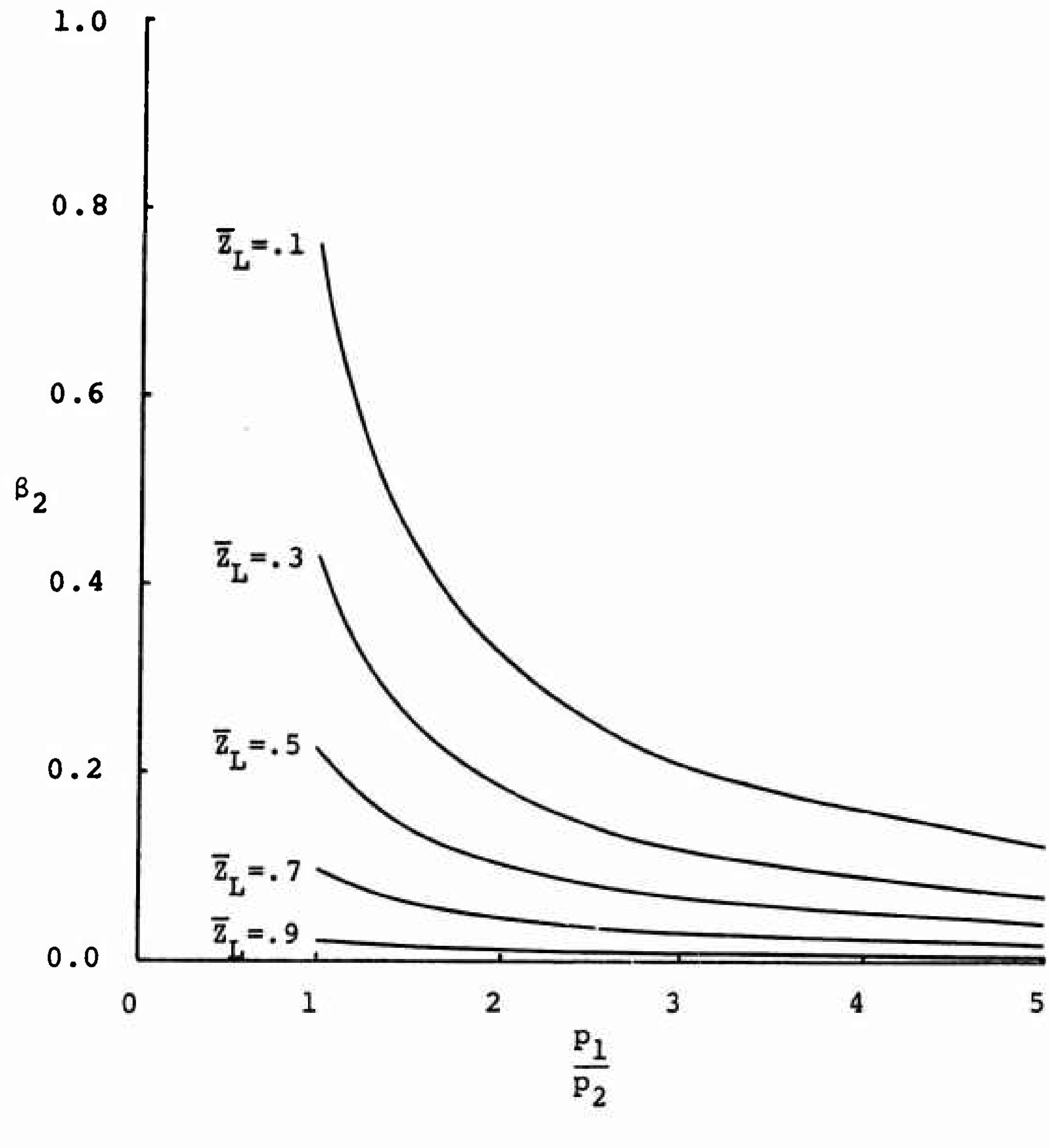

Figure 3.4c Dependence of SR-LSI Model $\beta_1 = 1$ Iso-$\bar{Z}_L$ Curve Endpoints on p_1/p_2

way a search increases the expected utility. If the posterior target location probability distribution results in the same target location guess as that implied by the prior distribution, the expected utility remains constant, which is zero on the normalized scale. That is, only search outcomes which change the target location guess increase the expected utility. If $p_1 = p_2 = .5$, any single contact results in $p'_{i*} > .5$ for the indicated target location guess, i*. If $p_1 > p_2$, only those search outcomes yielding $p'_2 > p_1$ contribute toward increasing the expected utility. As p_1 increases from a starting value of .5 such a shift rapidly requires increasing information quality.

The effect of deficiencies in search information can be offset by more effective response systems. This is seen by examining the tradeoff between these effects as represented by the parameters β_i and d for the SR-LSI model. Assume that $\beta_1 = \beta_2 = \beta$. A computer program was developed to compute iso-$\bar{Z}_L$ curves in the (β, d) plane. Since the expected utility is proportional to d, only one curve is computed for each prior probability distribution. The curves computed are those which yield absolute $\bar{Z}_L$ values of .5. The d value corresponding to a given β value for any other value of $\bar{Z}_L$ is

$$d = 2\,\bar{Z}_L\, d_{.5}(\beta),$$

where $d_{.5}(\beta)$ is the ordinate of the $\overline{Z}_L$ = .5 iso-$\overline{Z}_L$ curve at β.

Figure 3.5 shows these $\overline{Z}_L$ = .5 curves in the (β,d) plane for several values of p_1. The striking characteristic of these curves is that they exhibit increasing marginal utility. That is, better search systems (lower β) are accompanied by higher marginal savings in response effectiveness, d, per unit improvement in search system effectiveness, β, to attain the same expected utility value. This increasing marginal returns characteristic depends on the scales for the β and d parameters. If one is interested in economic tradeoffs, the relevant scales for measurement are the cost functions associated with β and d. Since $\beta = 0$ represents perfect reliability of the detector output (contacts), we presume that the cost associated with β increases with infinite slope as β approaches 0. Thus, the cost function of β is probably sufficiently nonlinear to dominate the increasing marginal utility property of the β vs d iso-$\overline{Z}_L$ curves. That is, an economic joint selection of β and d probably would not be a corner solution ($\beta = 0$ or $\beta = 1$) as is generally implied by increasing marginal utility tradeoff relationships.

Suppose that a search is planned on an intuitive basis without the benefit of a mathematical search allocation model. Then, the expected utility from the response pro-

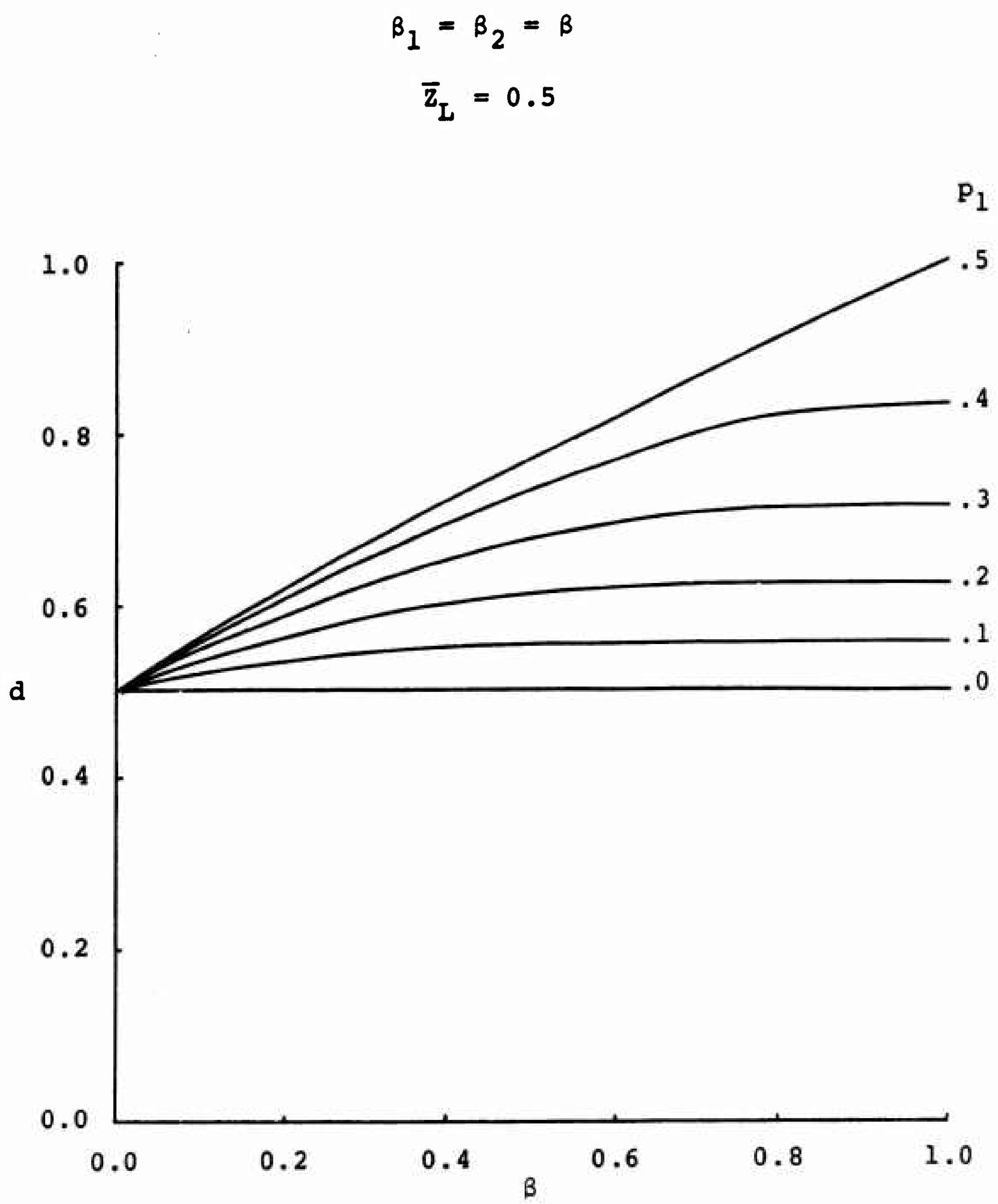

Figure 3.5 SR-LSI Model Iso-$\overline{Z}_L$ Curves in (β, d) Plane

cess will be sub-optimal by an amount which depends on the allocation. We investigate the robustness of the SR-LSI expected utility. A computer program was developed to examine the sensitivity of the SR-LSI expected utility to the allocation, (T_1,T_2), with $T_1 + T_2 = T$ fixed. Figures 3.6a to 3.6c show representative normalized Z_1 values as functions of T_1 for $T_1 + T_2 = \overline{T}$, the maximum amount of available search time used by an optimal allocation. Figure 3.6d shows Z_L as a function of T_1 (with $T_1+T_2 = T$) for different values of T. These plots reflect moderate to extreme sensitivity depending on the parameters. For sufficiently large T the expected utility decreases dramatically near $T_1 = 0$ or $T_2 = 0$. This sharp drop can be explained by considering the optimal target location guess plan for searches which are restricted to one box. The optimal guess plan is always of the form: Guess that the target is in box i unless a contact occurs in box (3-i). For large amounts of search time, the contact probability is high regardless of whether the target is in the box searched. Thus, the search provides negligible target location information for very high amounts of search time in one box. Hence, the normalized expected utility is zero for long duration searches confined to one box.

$\beta_1 = 0.01,\ k_1 = 1,\ p_1 = .23077$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .76923$

$\bar{T} = 8.8413$

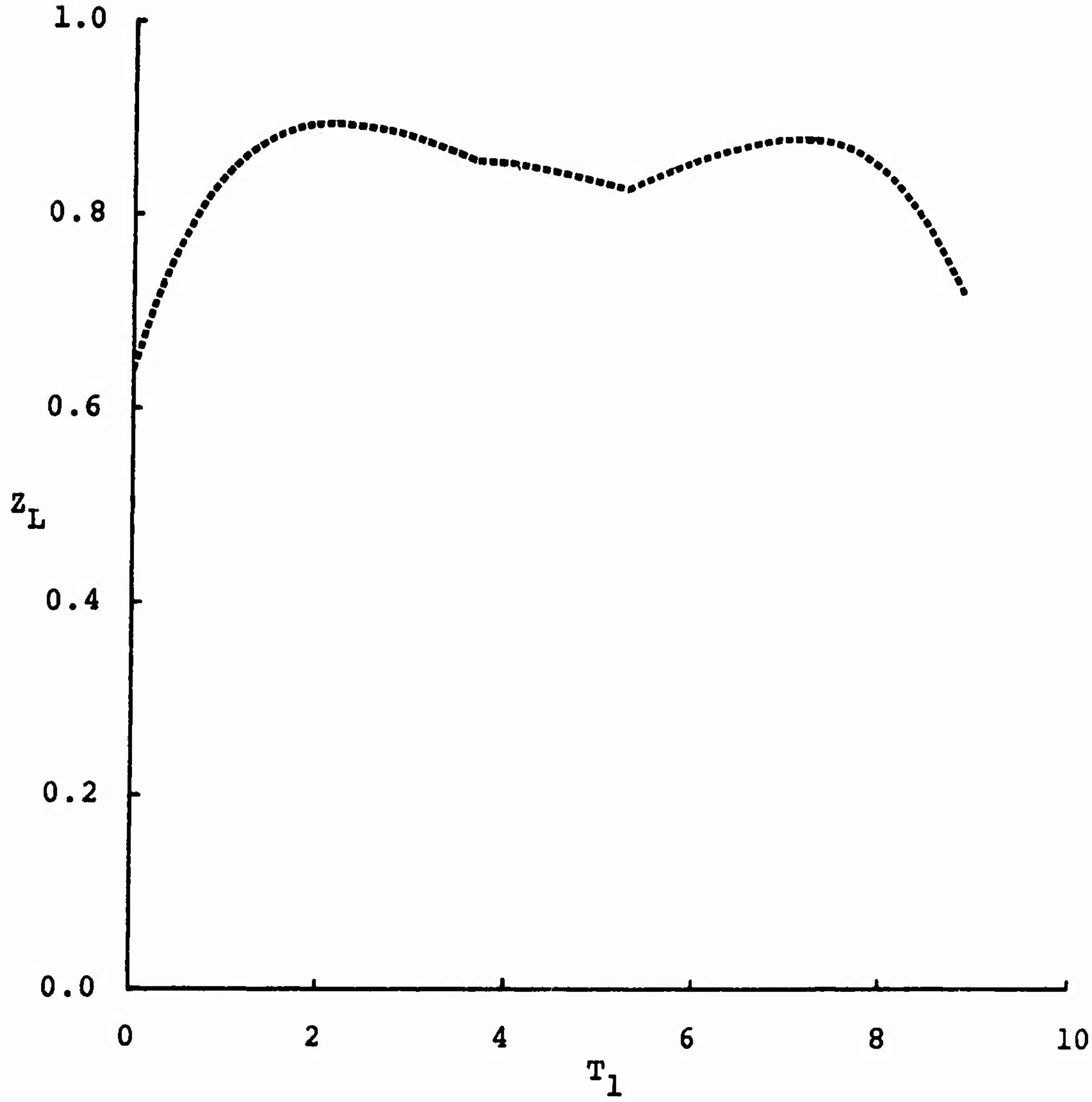

Figure 3.6a Normalized SR-LSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

$\beta_1 = 0.01,\ k_1 = 0.3,\ p_1 = 0.75$

$\beta_2 = 0.05,\ k_2 = 1.0,\ p_2 = 0.25$

$\bar{T} = 21.645$

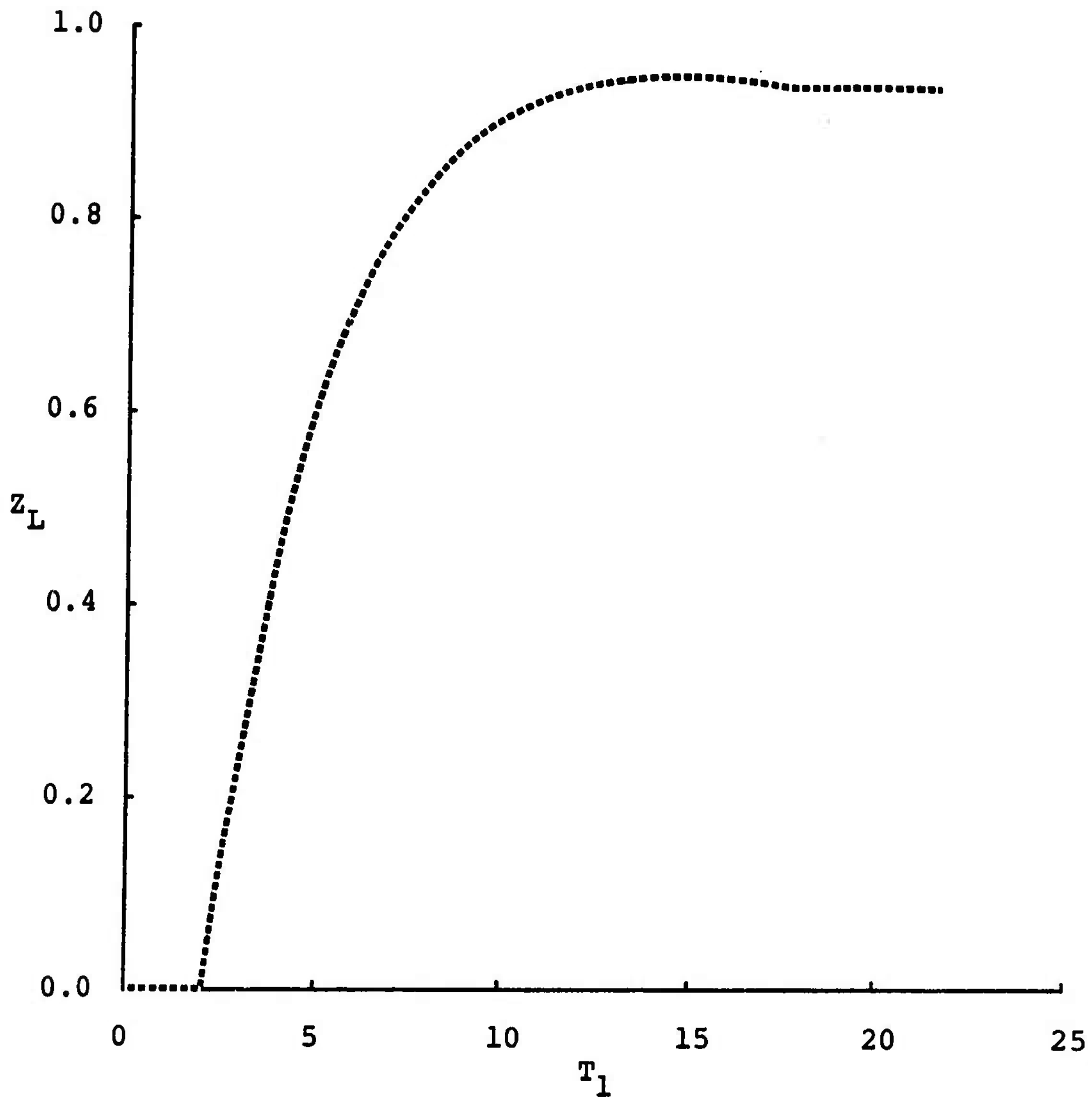

Figure 3.6b Normalized SR-LSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

$\beta_1 = 0.10,\ k_1 = 0.3,\ p_1 = .23077$

$\beta_2 = 0.05,\ k_2 = 1.0,\ p_2 = .76923$

$\bar{T} = 19.518$

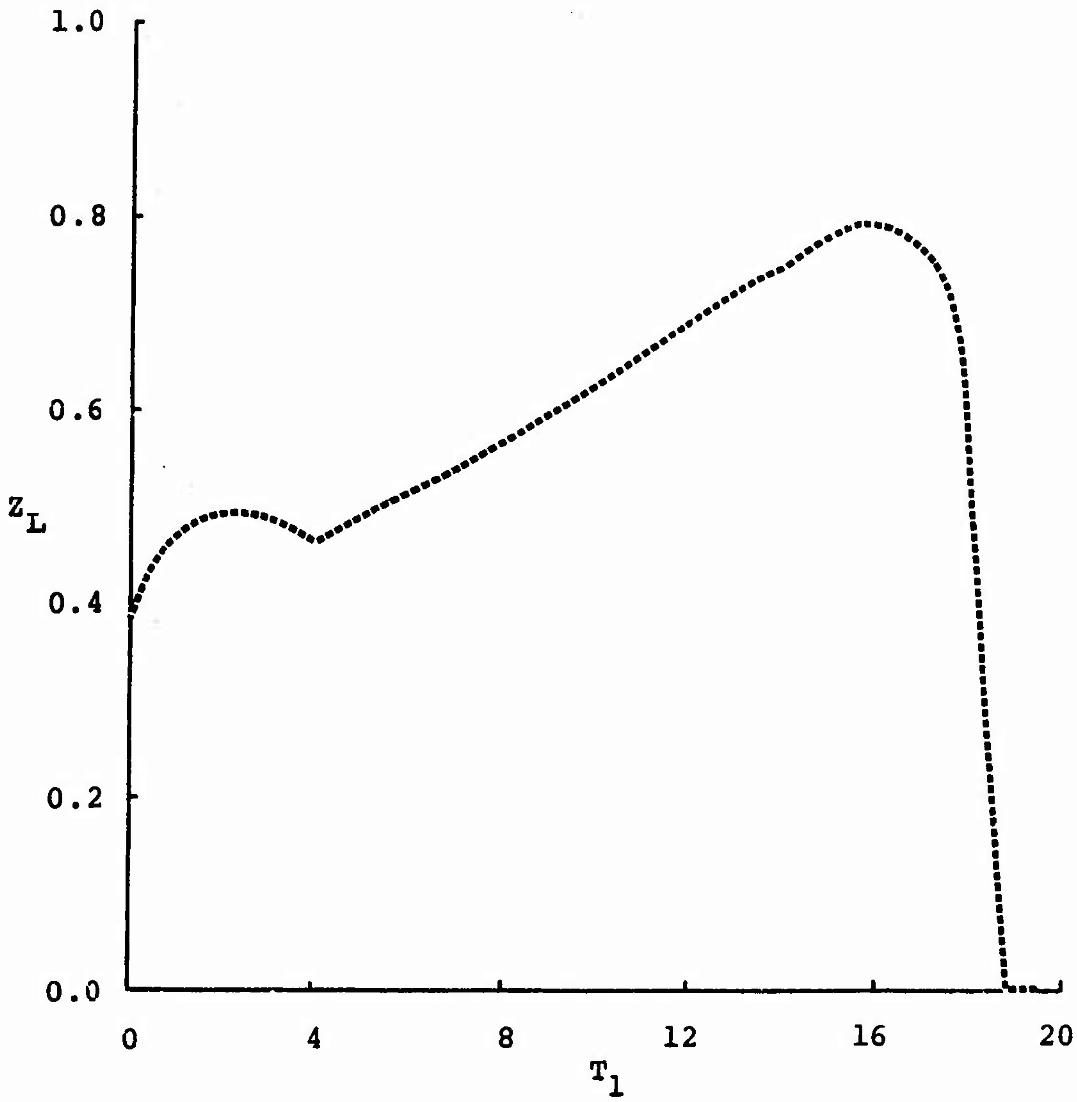

Figure 3.6c Normalized SR-LSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

$\beta_1 = 0.01,\ k_1 = 1,\ p_1 = .23077$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .76923$

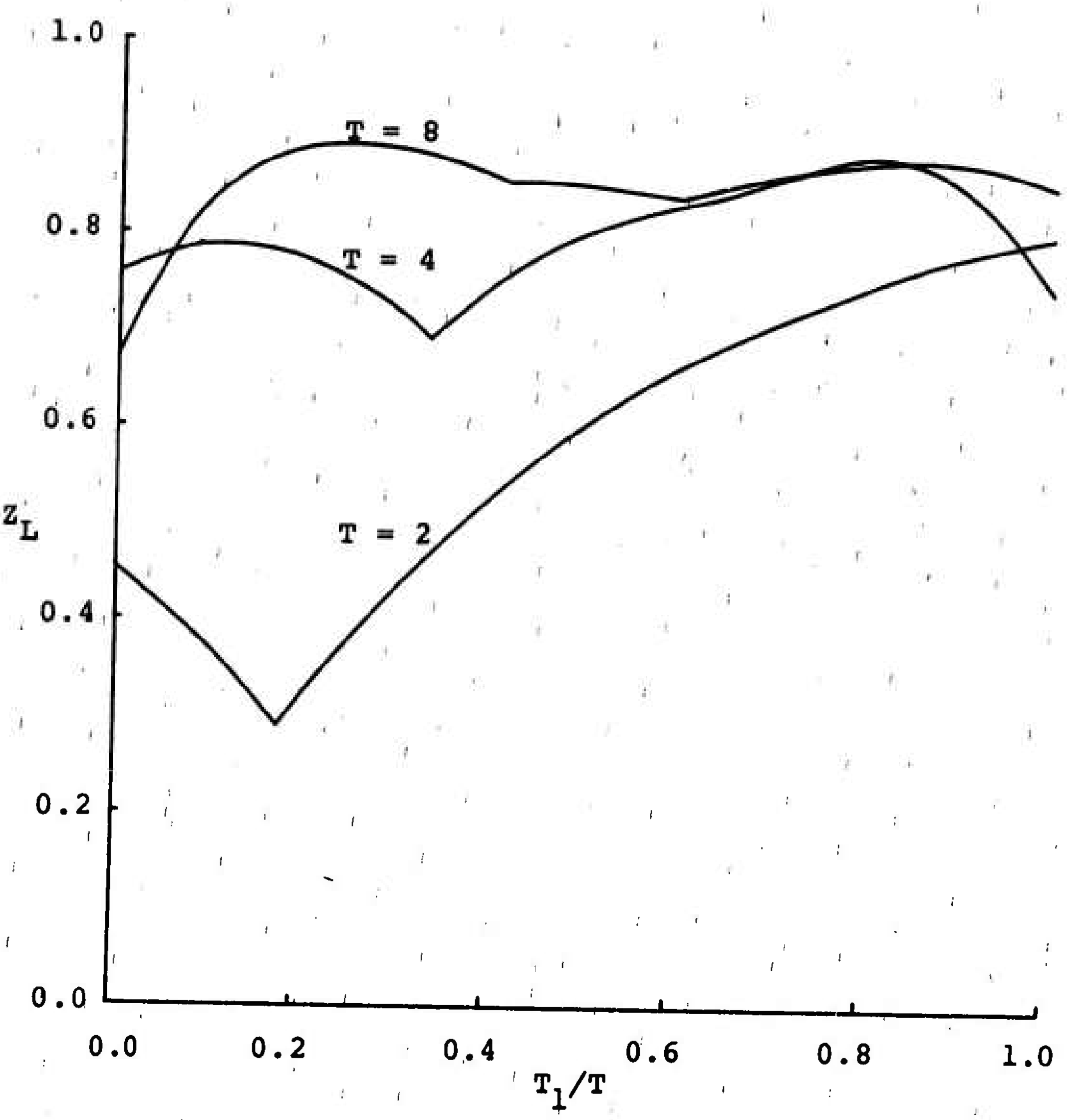

Figure 3.6d Normalized SR-LSI Model Expected Utility Sensitivity to Allocation Dependence on T

3.1.3 Summary of SR-LSI Results and Implications

SR-LSI optimal allocations as function of available search time, T, follow complicated optimal trajectories which may switch between two conditional trajectories as many as three times. As the available search time grows without bound both conditional trajectories terminate. Therefore, the optimal allocations, T_1^* and T_2^*, are bounded functions of the available search time (unless β_1 or β_2 is zero). The optimal allocations may be non-increasing functions of T with discontinuities at the switch times. This non-increasing character of the optimal allocations, T_1^* and T_2^*, implies that the optimal search plan can be computed only if T is known.

The optimal trajectories usually begin (for small T) along the coordinate axis corresponding to the box having the lower prior target location probability[1]. That is, for small T the optimal search is concentrated in the box where the target is least expected to be located. For moderate values of T the optimal allocation is usually along the conditional trajectory which is nearer to the T_i axis, where $k_i \geq k_j$, $j = 3-i$. For larger values of T the

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[1]This fails to hold only if the prior target location probability distribution completely determines the optimal guess for small T, i.e., there exist i and i = 3-i such that $\beta_i p_j > p_i$.

optimal allocation is usually along the conditional trajectory which is nearer to the T_i axis with $\beta_i \leq \beta_j$. And for most cases as T approaches $\overline{T}$ the optimal allocation is along the conditional trajectory which is nearer to the T_j axis with $\beta_i \leq \beta_j$.

For most cases examined the available search time is not shared approximately evenly ($T_1^* = T_2^*$) for any amount of available search time, T. In particular, if the parameters for the two boxes are identical, $T_1^* = T_2^*$ is never optimal. The ratio T_1^*/T_2^* for a wide variety of identical box cases was never found to be between 0.37 and 2.69. Thus, in the event that the decision maker does not know the parameter values precisely[1], he probably should avoid any impulse to deal with his uncertainty by allocating equal amounts of time to each of the two boxes.

The decrease in attainable expected utility value attributable to the false contacts (false contact loss) increases as a function of T for small T. This increasing

---

[1]Since the SR-LSI expected utility is linear in p_1, if one has a prior probability distribution for the prior probability p_1 with the other parameters known, the optimal allocation is found by using the expected value of p_1 in the solution given in Appendix B. If the ratios p_1/p_2 and k_1/k_2 are approximately one, the ratios T_1^*/T_2^* have been found to be significantly different from one for the cases examined.

character of the false contact loss may continue for moderate and large values of T or the false contact loss may exhibit alternating intervals for which it first increases and then decreases as a function of T. Such complex increasing then decreasing character stems directly from the discontinuities in the optimal expected utility slope associated with the switching of the optimal trajectory between the two conditional trajectories. Also associated with these objective function slope discontinuities are intervals of planned search time, T, which cannot be optimal if there is a fixed charge proportional to the planned search time.

The decrease in SR-LSI expected utility from using a Koopman model rather than the optimal allocation (the Koopman loss) is a complicated function of T. For most cases the Koopman loss grows with increasing T for small T. For T approximately three to four times the expected time to contact for the Koopman allocation, the Koopman loss reaches a relative minimum which is usually small. For moderate T the Koopman loss may alternate from increasing to decreasing due to the discontinuities in the optimal SR-LSI objective function slope. For large T, the Koopman loss increases monotonically approaching the normalized value $\overline{Z}_L$ asymptotically as T grows without bound. This implies that the search should be limited to

a modest upper bound on the total planned searching time.

The contribution of the search to the SR-LSI expected utility is greatest if $p_1 = p_2$. But, the absolute value of the optimal expected utility is maximized by $p_1 = 0$ or 1. In fact, the optimal expected utility for small T is minimized by $p_1 = p_2$. From the computations performed it appears that the optimal expected utility is minimized by $p_1 = p_2$ for any amount of available search time. That is, the search contribution is greatest when the prior target location probability distribution is that of complete ignorance (maximum entropy).

If $\beta_1 = \beta_2 = 0$ (no false contacts), the optimal search is concentrated in one of the two boxes. For the cases with β_1 and β_2 positive, however, the optimal search may require that positive amounts of search time be allocated to both boxes. For all cases examined there exists a "one box" search allocation with either $T_1 = 0$ or $T_2 = 0$ which yields values of SR-LSI expected utility which are nearly optimal. That is, for practical purposes, it is unnecessary to split the search between the two boxes.

3.2 Complete Search Information (SCI) SR Model

The LSI version analysis of the decision maker's search allocation problem is appropriate only if the times at which contacts were made are unavailable as inputs to

the target location guess decision. If the contact times are available, the guess decision can profitably include this data as well as the number of contacts in each box. In Appendix D we discuss an algorithm for solving the decision maker's search allocation problem given the times-to-contact. This algorithm was derived to solve the following CSI formulation of the problem:

Let

t_i = amount of search time used in box i when either a contact occurs or the search is terminated for lack of additional search time,

T_i = amount of available search time allocated to box i,

$\underline{t}$ = the vector (t_1,t_2),

$\underline{T}$ = the vector (T_1,T_2),

$p_i'(\underline{t},\underline{T})$ = Pr(Target is in box i | $\underline{t},\underline{T}$),

$$u(\underline{t},\underline{T}) = \frac{p_1'(\underline{t},\underline{T})}{p_2'(\underline{t},\underline{T})} ,$$

$$G(\underline{t},\underline{T}) = \{g \mid p_g'(\underline{t},\underline{T}) = \max_i\{p_i'(\underline{t},\underline{T})\}\}.$$

$\underline{T}$ is the search plan; $\underline{t}$ is the search outcome.

Clearly, the optimal target location guess policy is given by:[1]

---

[1]This follows from the symmetry of the reward for correctly guessing the target location. The extension to an assymmetric structure is straightforward.

If $u(\underline{t},\underline{T}) > 1$, $G(\underline{t},\underline{T}) = \{1\}$,

if $u(\underline{t},\underline{T}) = 1$, $G(\underline{t},\underline{T}) = \{1,2\}$,

if $u(\underline{t},\underline{T}) < 1$, $G(\underline{t},\underline{T}) = \{2\}$.

And the following are immediate consequences of Bayes' theorem.

$$u(\underline{t},\underline{T}) = \frac{p_1 e^{-(1-\beta_1)k_1T_1}}{p_2 e^{-(1-\beta_2)k_2T_2}} \quad \text{for } t_1 = T_1,\ t_2 = T_2 \quad (24a)$$

$$= \frac{p_1 e^{-(1-\beta_1)k_1T_1}}{\beta_1 p_2 e^{-(1-\beta_2)k_2T_2}} \quad \text{for } t_1 < T_1,\ t_2 = T_2 \quad (24b)$$

$$= \frac{\beta_2 p_1 e^{-(1-\beta_1)k_1T_1}}{p_2 e^{-(1-\beta_2)k_2T_2}} \quad \text{for } t_1 = T_1,\ t_2 < T_2 \quad (24c)$$

$$= \frac{\beta_2 p_1 e^{-(1-\beta_1)k_1T_1}}{\beta_1 p_2 e^{-(1-\beta_2)k_2T_2}} \quad \text{for } t_1 < T_1,\ t_2 < T_2. \quad (24d)$$

The case with β_1 and β_2 zero may be ignored since the solution for this special case is identical to that of the SR-LSI model. The complementary probability distribution function for $\underline{t}$ is

$$\overline{F}(\underline{t}) = \begin{cases} p_1 e^{-k_1 t_1 - \beta_2 k_2 t_2} + p_2 e^{-\beta_1 k_1 t_1 - k_2 t_2} & \text{for } \underline{0} \leq \underline{t} < \underline{T} \\ 0 & \text{for } \underline{t} \nless \underline{T} \end{cases} \quad (25)$$

Therefore, the constrained search allocation problem is

$$\underset{\underline{T}}{\text{Maximize}}\ Z = \iint_{\underline{0} \leq \underline{t} \leq \underline{T}} \max_{i=1,2} \{p_i^!(\underline{t},\underline{T})\}\ dF(\underline{t})$$

$$\text{Subject to } T_1,\ T_2 \geq 0,\ T_1 + T_2 \leq T. \quad (26)$$

<u>Solution for β_1, $\beta_2 > 0$</u>

In Appendix D this problem is solved by comparing the objective function, Z, to the corresponding function, $Z(\infty)$, with $T_1 = T_2 = \infty$. Let

$$H(\underline{T}) = Z(\infty) - Z.$$

It is shown in Appendix D that $H(\underline{T})$ is minimized over $T_1 + T_2 = T$ by some point between two straight lines, which we will call lines b and c,

$$s(\underline{T}) = \ln \frac{p_1}{\beta_i p_2} \equiv b, \quad (27)$$

and

$$s(\underline{T}) = \ln \frac{\beta_2 p_1}{p_2} \equiv c. \quad (28)$$

where

$$s(\underline{T}) = (1-\beta_1)k_1 T_1 - (1-\beta_2)k_2 T_2.$$

Lines b and c for a representative case are shown in Figure 3.7. Along any line $s(\underline{T}) = e$ with $c \leq e \leq b$, $H(\underline{T})$ is a non-increasing function of T_1+T_2. Further, for large $T_1 + T_2$, $H(\underline{T})$ decreases exponentially to zero along the line $s(\underline{T}) = e$ with the rate parameter of the exponential form independent of e. Therefore, the optimal allocation, (T_1^*,T_2^*), for large T lies along a straight line $s(\underline{T}) = e$ with $T_1^*+T_2^* = T$ as $T \to \infty$.

The optimal trajectory (plot of (T_1^*,T_2^*) with T as a parameter) consists of alternating segments of two candidate conditional trajectories. The coordinates of points along these two conditional trajectories can be computed as functions of roots of the functions which express the first order marginal conditions for minimizing $H(\underline{T})$ along $T_1+T_2 = T$.

3.2.1 Comparison of SR-CSI Model with SR-LSI and Koopman Models

Based on the analysis described in Appendix D, a Fortran IV computer program was coded for the Pentagon IBM 7094 to compute approximate numerical solutions to the CSI version of the decision maker's search allocation problem.

Figures 3.8a to 3.8 d depict the CSI optimal trajectories for the same cases as depicted by Figures 3.1a to 3.1d for the LSI version of the SR model. For small T, the optimal allocations are identical for the LSI and CSI versions of the SR model This is the case because for

$\beta_1 = 0.1,\ k_1 = 1,\ p_1 = 0.5$

$\beta_2 = 0.5,\ k_2 = 1,\ p_2 = 0.5$

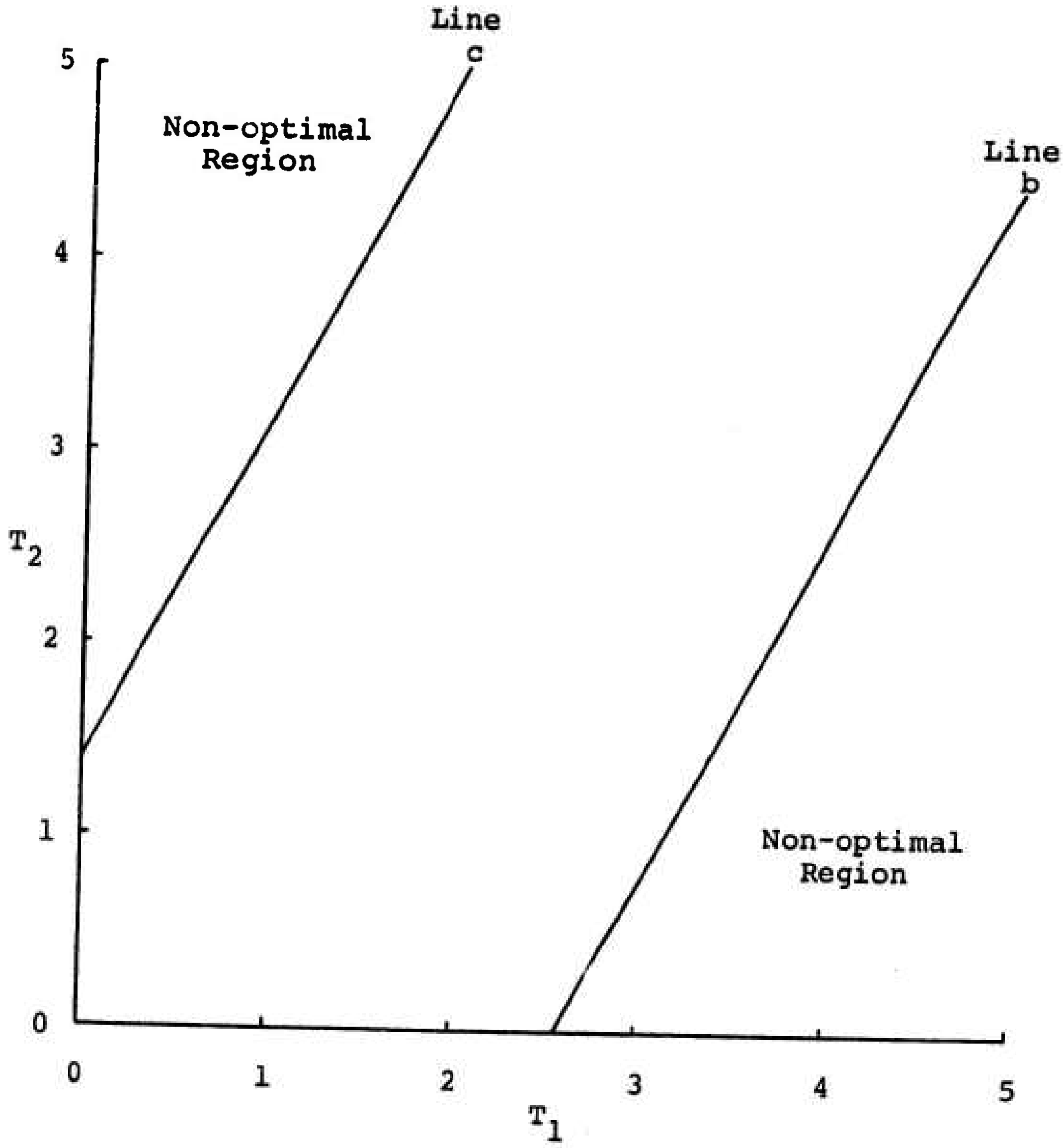

Figure 3.7 Bounds for SR-CSI Model Optimal Allocation

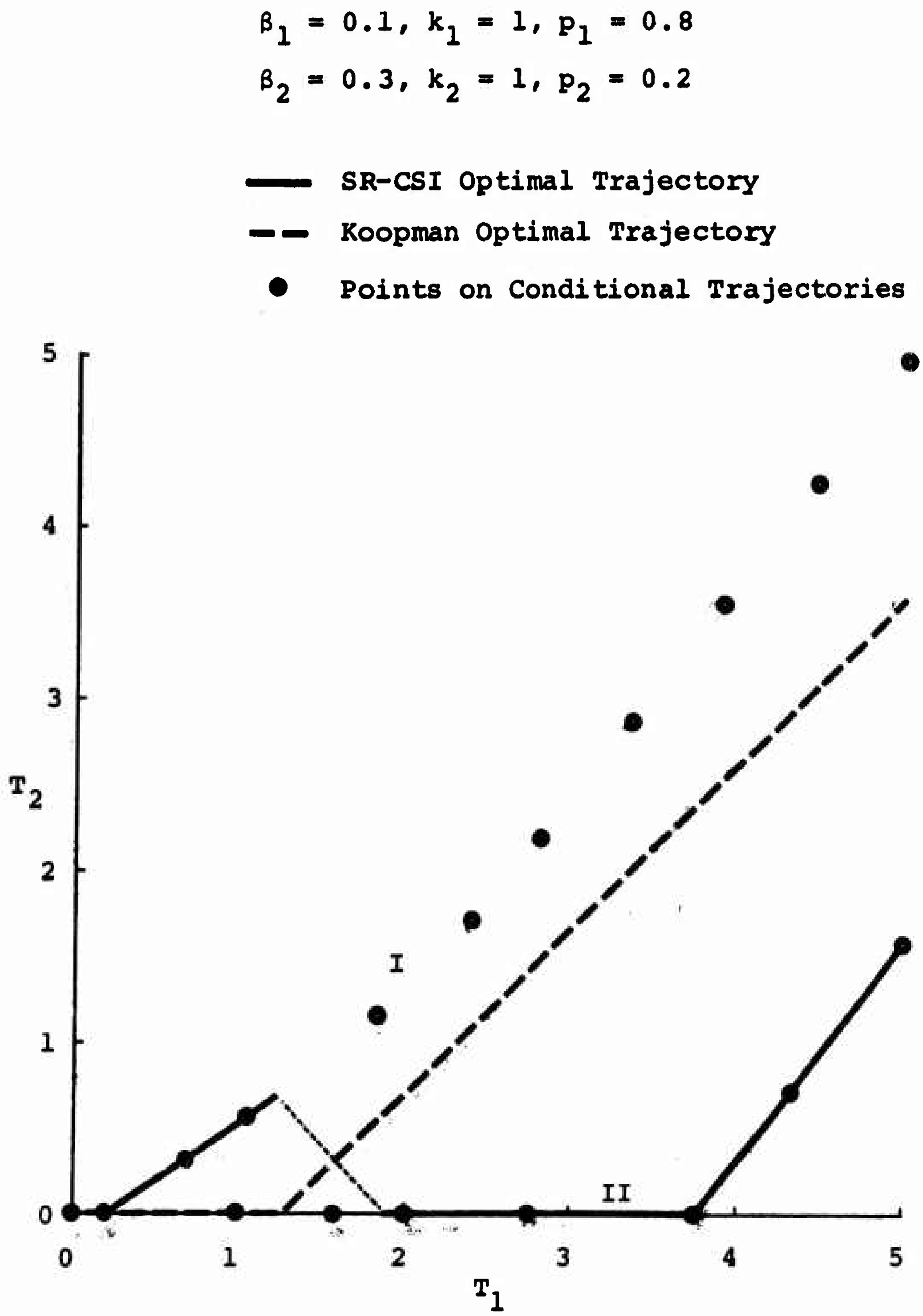

Figure 3.8a SR-CSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.01,\ k_1 = 0.5,\ p_1 = .47368$

$\beta_2 = 0.10,\ k_2 = 1.0,\ p_2 = .52632$

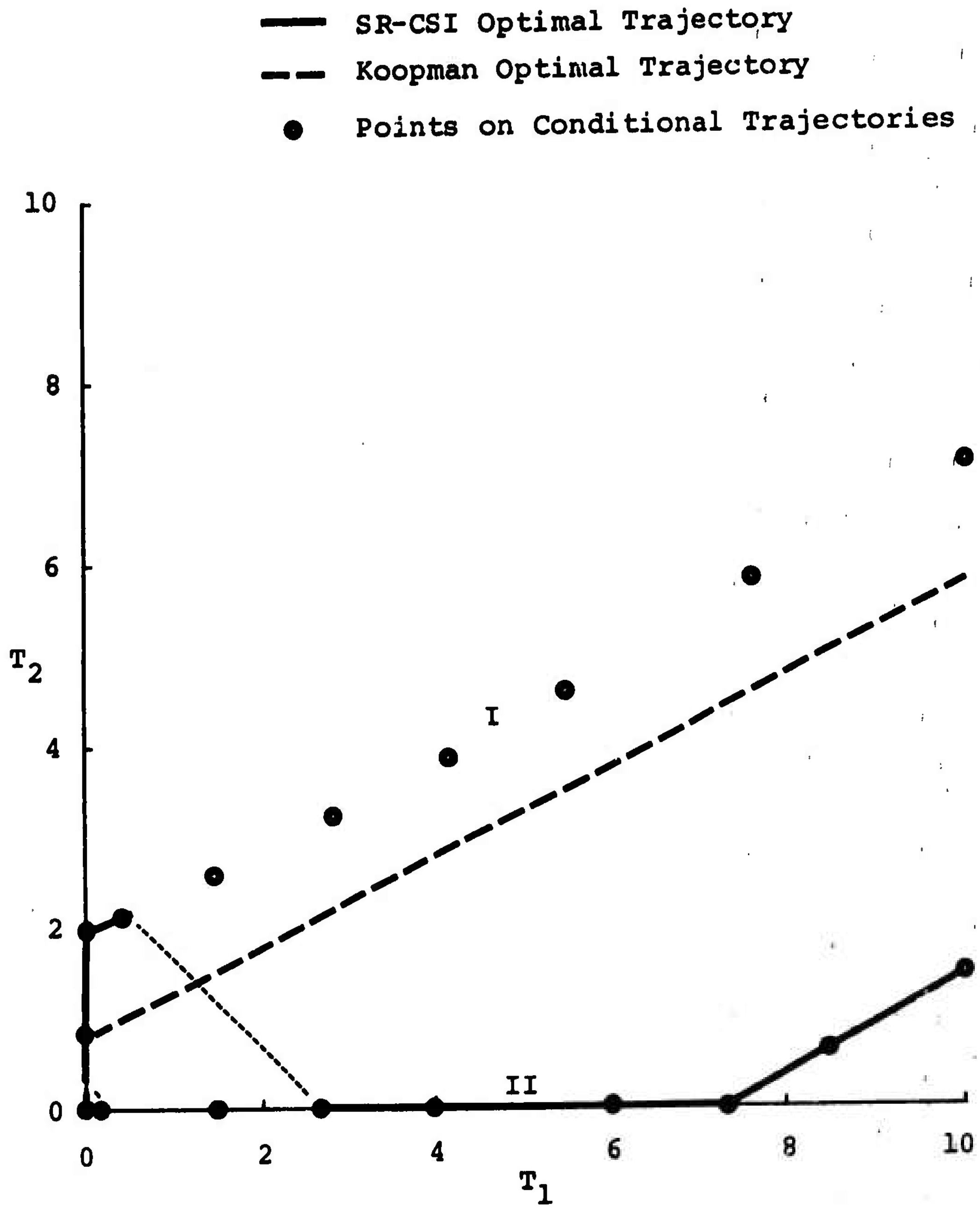

Figure 3.8b SR-CSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.10,\ k_1 = 1,\ p_1 = .47368$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .52632$

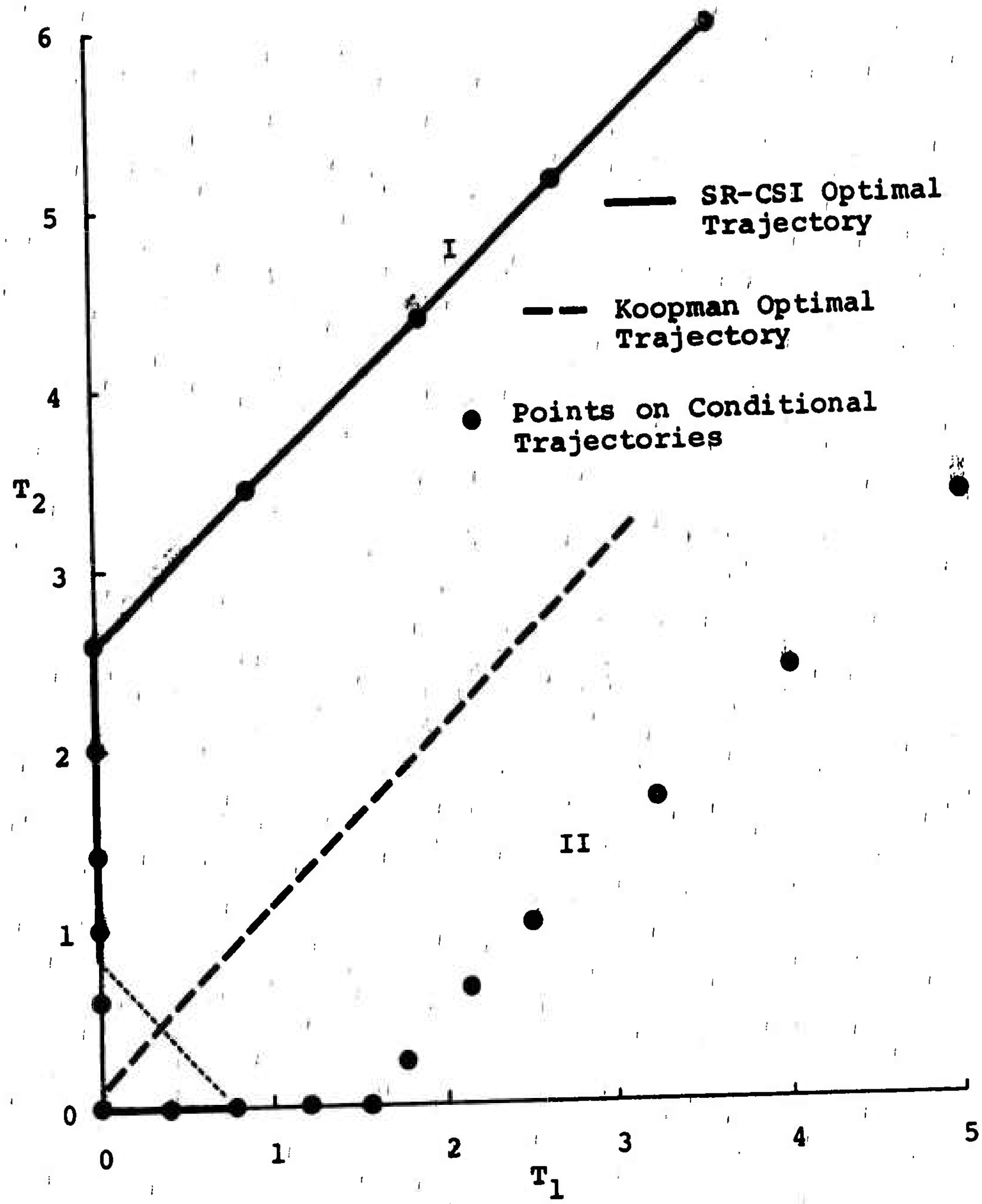

Figure 3.8c SR-CSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.1,\ k_1 = 1,\ p_1 = 0.5$

$\beta_2 = 0.5,\ k_2 = 1,\ p_2 = 0.5$

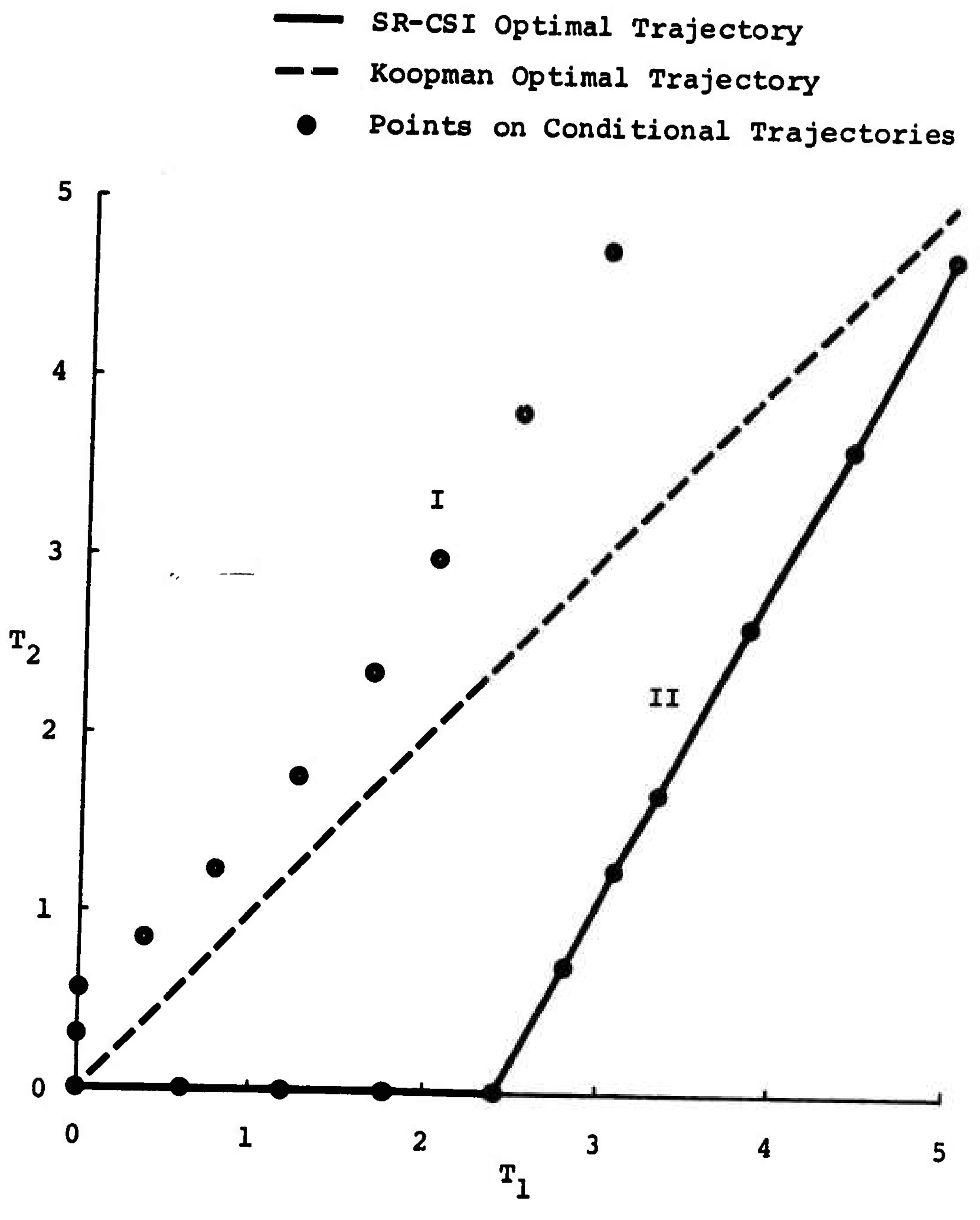

Figure 3.8d SR-CSI Model and Koopman Optimal Trajectories

optimal CSI allocations of small T, the optimal target location guess, given any possible search outcome ($\underline{t}$), is the same as the optimal guess based only on the numbers of contacts for the two boxes. That is, for small T the numbers of contacts for the two boxes are sufficient to make the target location guess decision.

As T increases the SR-CSI optimal trajectory may switch between two conditional trajectories in a manner similar to the SR-LSI optimal trajectory switching. For most of the cases examined the number of switches in the SR-CSI optimal trajectory has been one less than the number of switches in the corresponding SR-LSI optimal trajectory. Exceptions to this relationship have been cases in which both the SR-LSI and SR-CSI optimal trajectories contained either one switch or no switches. Because most of the SR-CSI optimal trajectories contain fewer switches than the SR-LSI optimal trajectories, the amount of information regarding the value of T needed to determine an SR-CSI optimal search plan is frequently less than that required to determine the corresponding SR-LSI optimal search plan.

In contrast to the SR-LSI conditional trajectories, the two SR-CSI conditional trajectories are unbounded lying along straight lines of the same positive slope as T increases without bound. The portions of the SR-CSI

conditional trajectories which connect the terminal straight line portions with the origin coincide with the corresponding portions of the SR-LSI conditional models. This coincidence of initial portions of SR-LSI and SR-CSI conditional trajectories stems directly from the sufficiency of the numbers of contacts in the two boxes for the target location guess decision. The unboundedness of the SR-CSI conditional trajectories implies that the optimal SR-CSI search is unbounded as T increases without bound. Thus, the optimal SR-CSI search plan always allocates all of the available search time to the two boxes.

Corresponding to the cases depicted in Figures 3.8a to 3.8d, Figures 3.9a to 3.9d show the normalized SR-CSI expected utility values, Z_C, as functions of T for the optimal SR-CSI search allocation and the Koopman allocation. Also shown in these figures are the "perfect detector" expected utility functions which could be achieved by a "perfect detector" system having the same true detection rates as implied by the parameter values $(k_i - \beta_i k_i)$, but no false contacts. The optimal SR-CSI expected utility curves coincide with the corresponding optimal SR-LSI expected utility curves for the initial intervals for which the SR-LSI and SR-CSI optimal trajectories coincide. These initial intervals have been found to be the same order of magnitude as the expected time-to-contact for

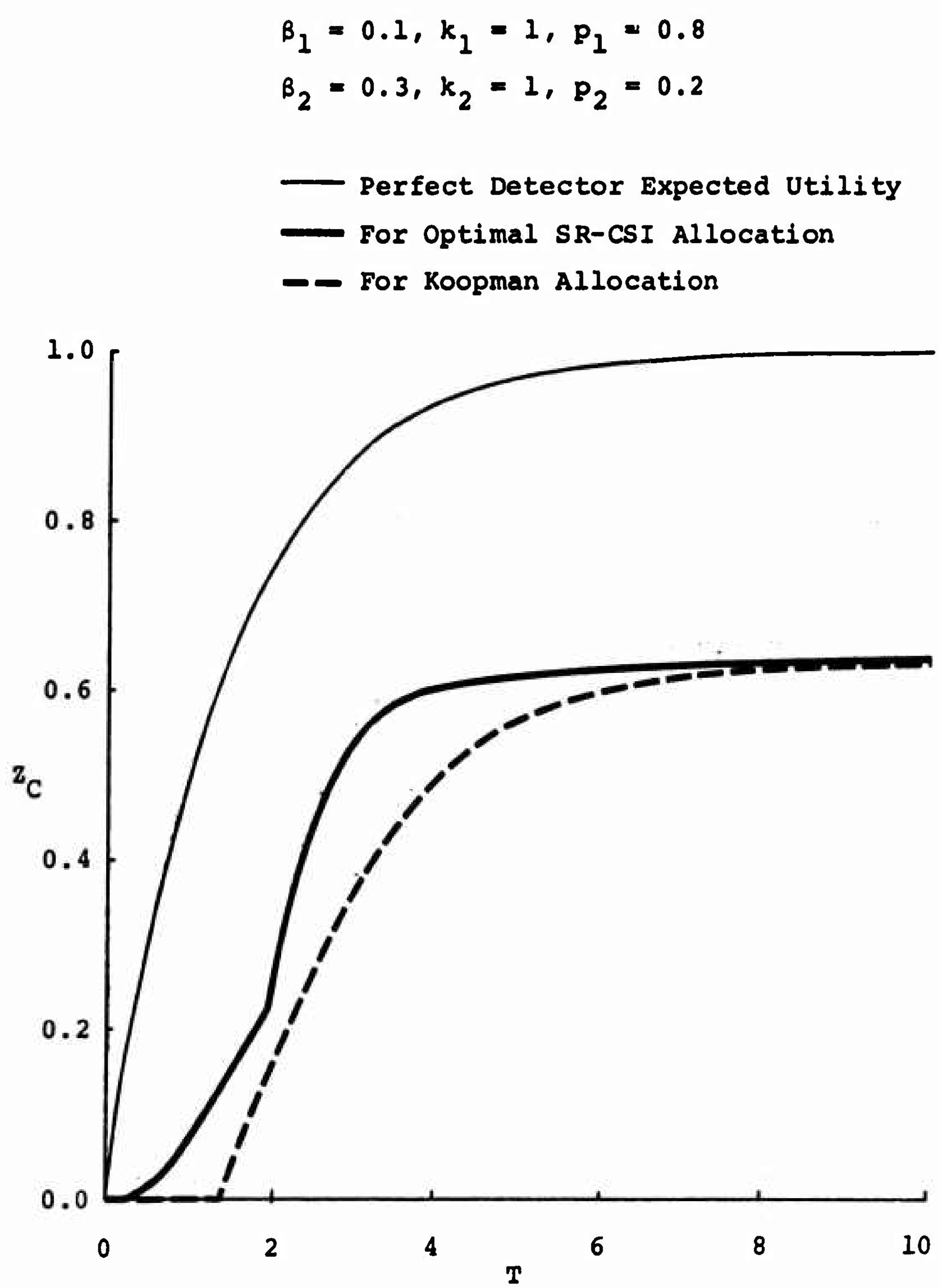

Figure 3.9a Normalized SR-CSI Model Expected Utility vs Available Search Time

$\beta_1 = 0.01,\ k_1 = 0.5,\ p_1 = .47368$

$\beta_2 = 0.10,\ k_2 = 1.0,\ p_2 = .52632$

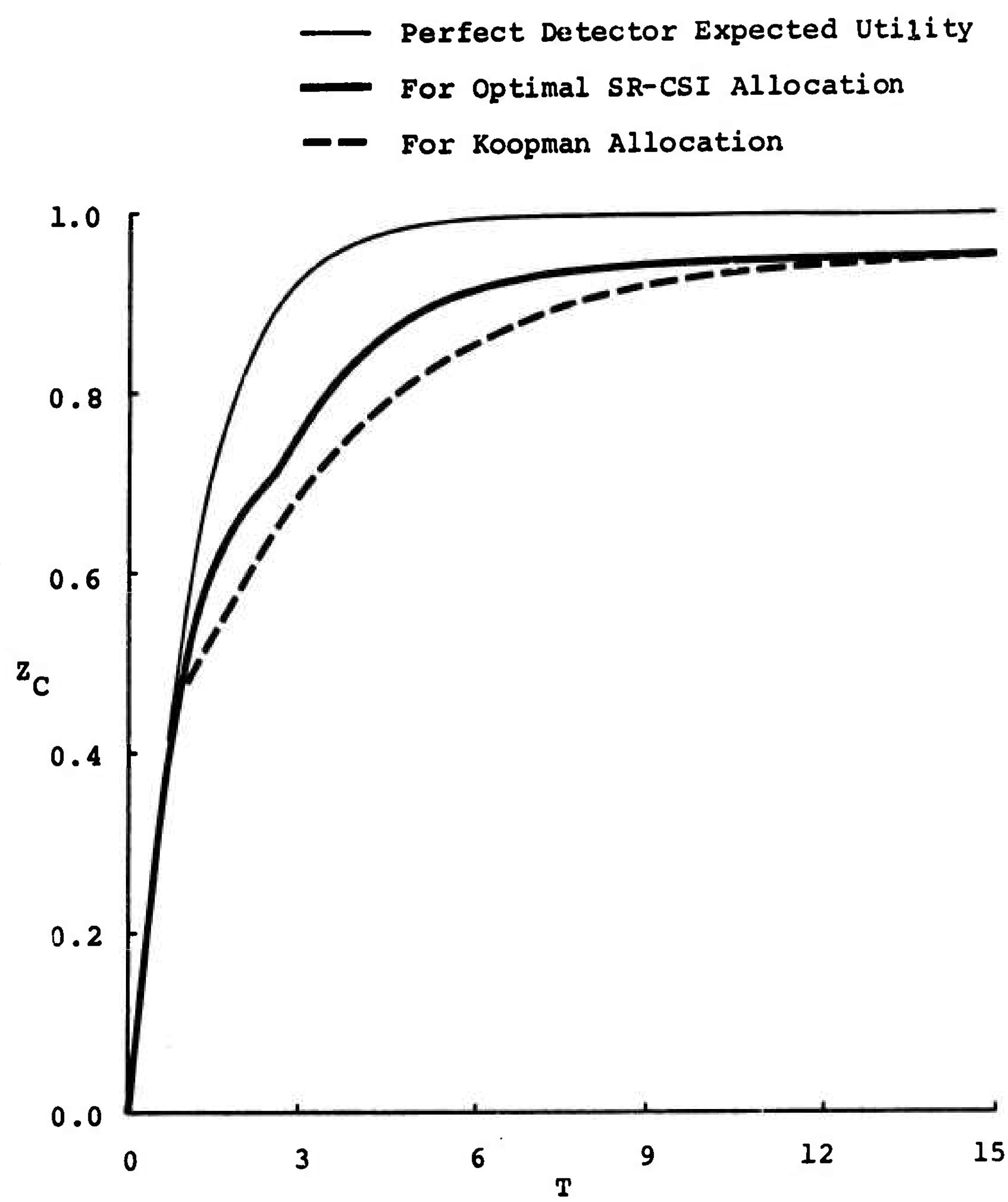

Figure Normalized SR-CSI Model Expected Utility vs Available Search Time

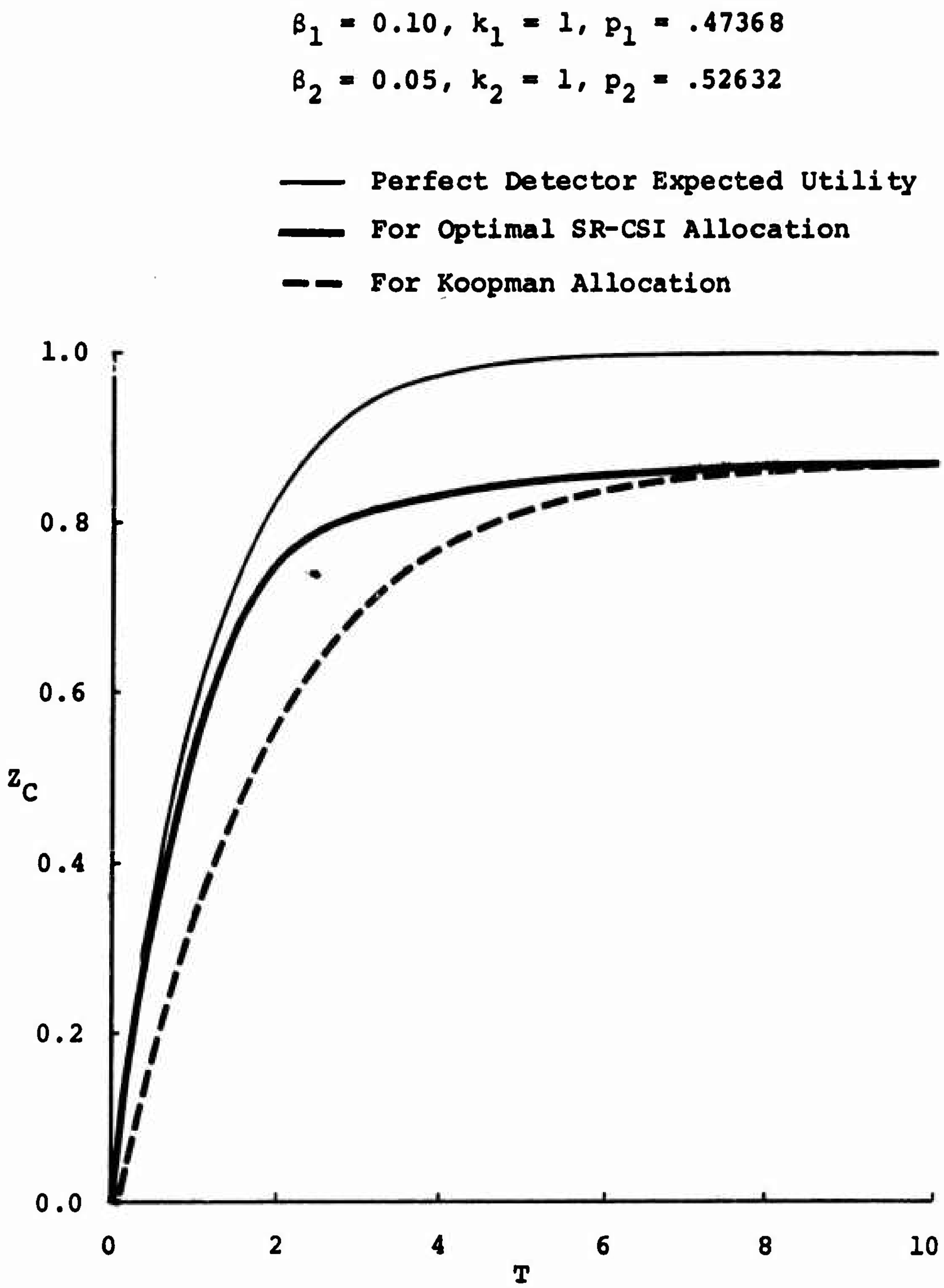

Figure 3.9c Normalized SR-CSI Model Expected Utility vs Available Search Time

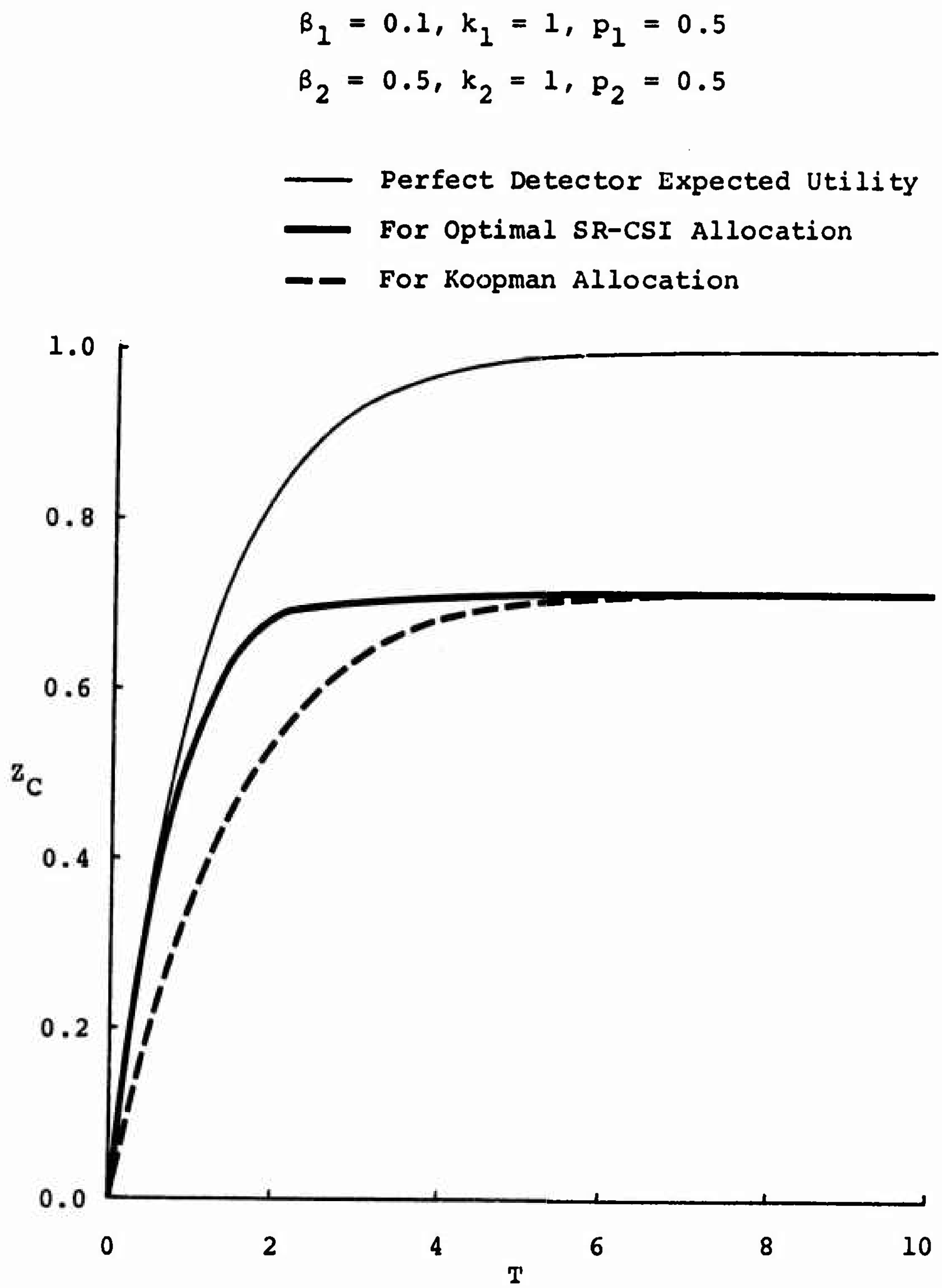

Figure 3.9d Normalized SR-CSI Model Expected Utility vs Available Search Time

the Koopman model. Although the SR-CSI optimal expected utility values are higher than the corresponding SR-LSI optimal expected utility values for larger values of T, the difference between these two curves has been rather small (less than three percentage points on the normalized SR expected utility scale) for all cases examined for all T. Thus, the use of search contact times in making the target location guess decision permits at most a small gain in optimal expected utility compared with basing the guess on the numbers of contacts in the two boxes.

Since the optimal SR-CSI expected utility curves are substantially the same as the SR-LSI curves, the expected utility cost of the false contacts is substantially the same for the SR-CSI and SR-LSI models. In contrast, the expected utility cost associated with the use of the Koopman search plan is not the same for these two models. For small and moderate values of T these sets of curves are substantially identical. But as T grows large the SR-CSI expected utility curves for the Koopman search allocations asymptotically approach the optimal objective function curves while the SR-LSI curves for the Koopman allocations drop asymptotically to the normalized expected utility value zero. That is, if the contact times are used in making the target location guess decision, there is no danger of planning to use too much search time.

3.2.2 Sensitivity of SR-CSI Results

Behavior for Small T

Since the LSI and CSI versions yield identical results for small T, the sensitivity results derived in Section 3.1.2 for the LSI version for small T hold for the CSI version as well.

Behavior for Large T

A computer program was developed to compute traces in the (β_1,β_2) plane corresponding to constant limiting normalized values of the SR-CSI expected utility, $\overline{Z}_C$, as T increases. Figures 3.10a to 3.10c show these iso-$\overline{Z}_C$ curves for the same cases as depicted in Figures 3.3a to 3.3c for the SR-LSI version. These two families of curves are similar having identical endpoints and general shape. But the CSI curves are more gently curved. Thus, if neither β_i is approximately equal to one, the CSI version achieves a moderately higher value of $\overline{Z}_C$ than the $\overline{Z}_L$ of the LSI version. Thus the best one box search yields expected utility values for large T which are lower than the optimal expected utility by amounts which are slightly greater than the corresponding expected utility loss for the best one box search for the SR-LSI model.

To investigate the tradeoff between search system and response system parameters, assume that $\beta_1 = \beta_2 = \beta$.

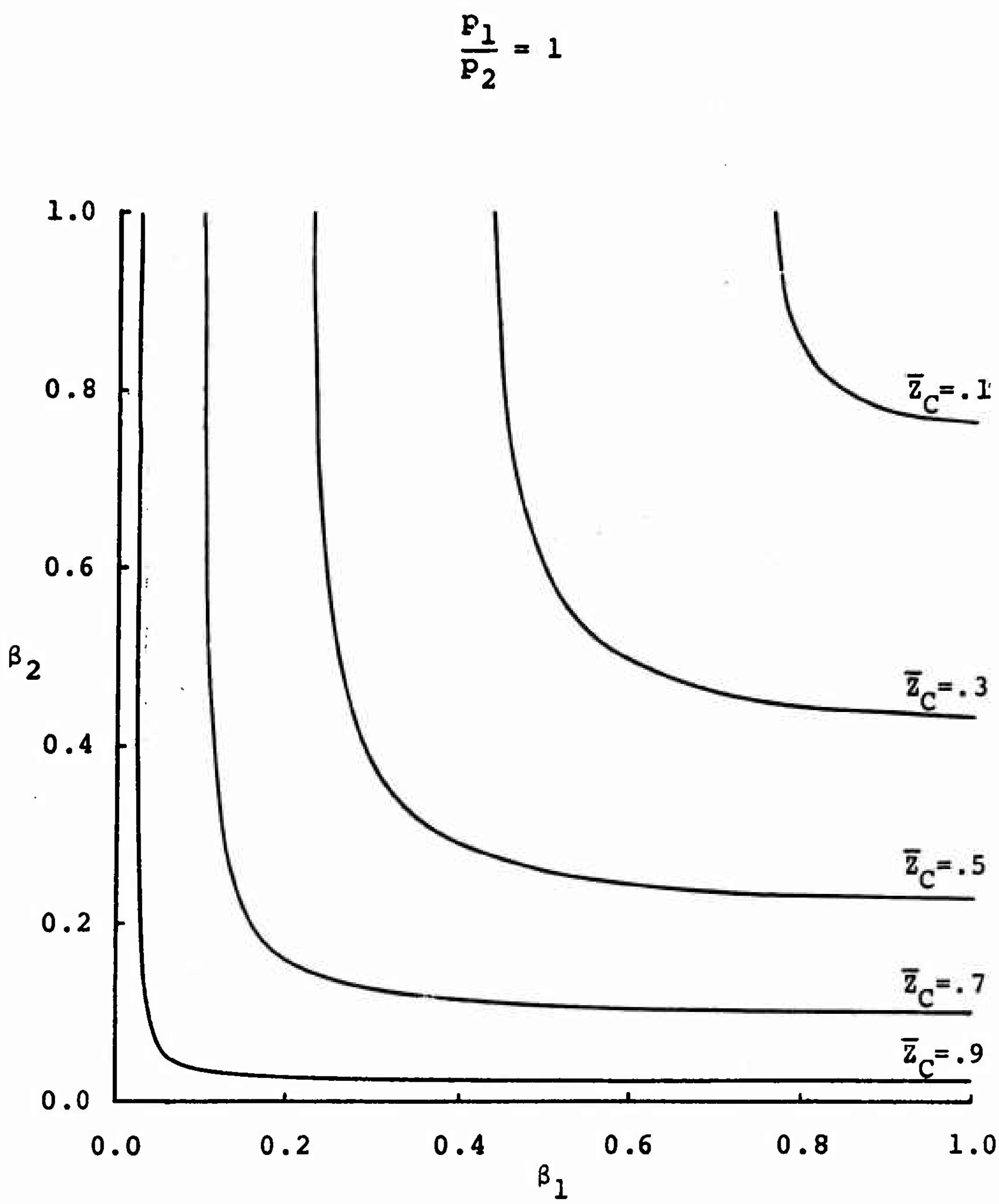

Figure 3.10a SR-CSI Model Iso-$\bar{Z}_C$ Curves in (β_1, β_2) Plane

$$\frac{P_1}{P_2} = 1.1$$

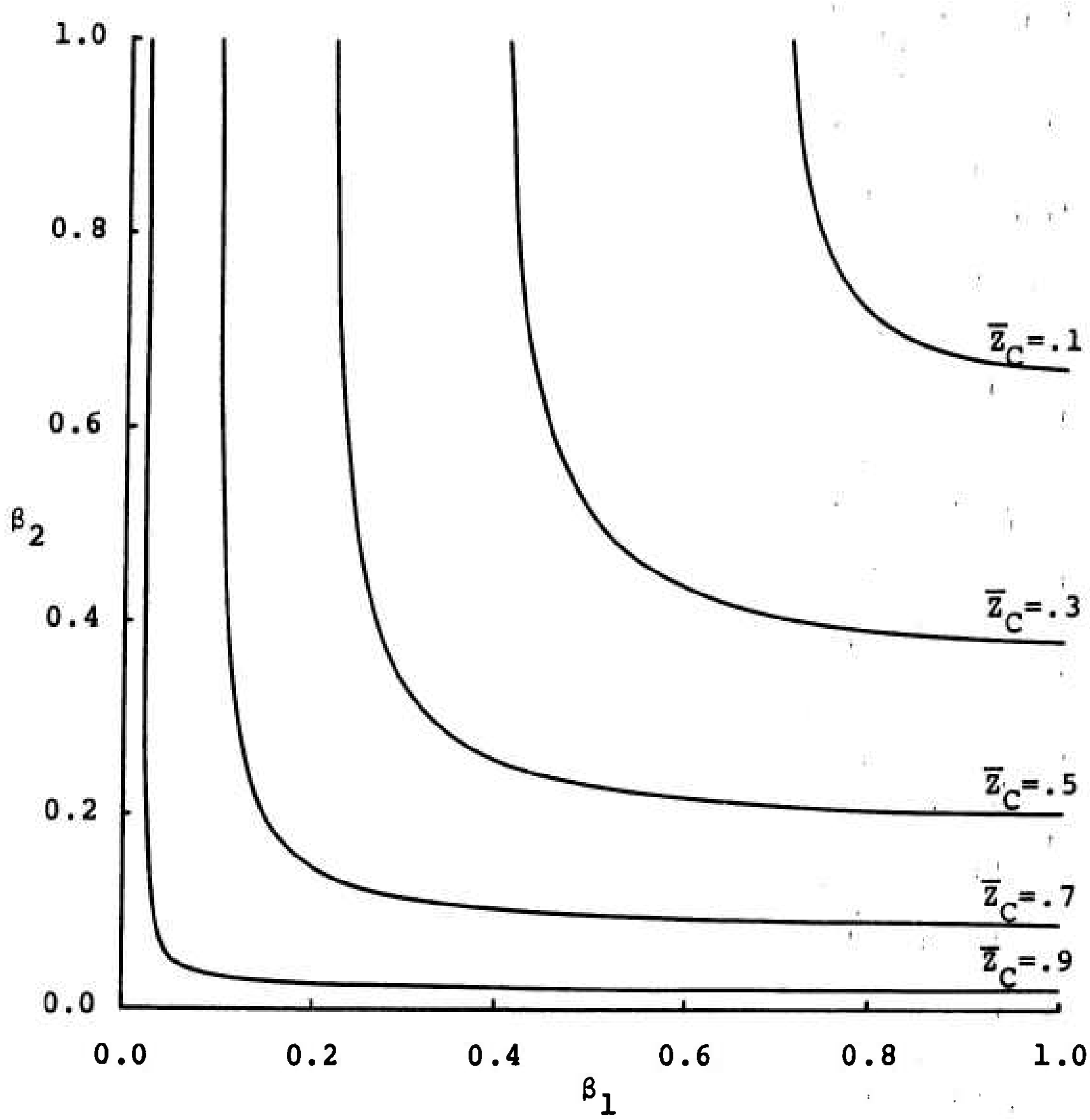

Figure 3.10b SR-CSI Model Iso-$\overline{Z}_C$ Curves in (β_1, β_2) Plane

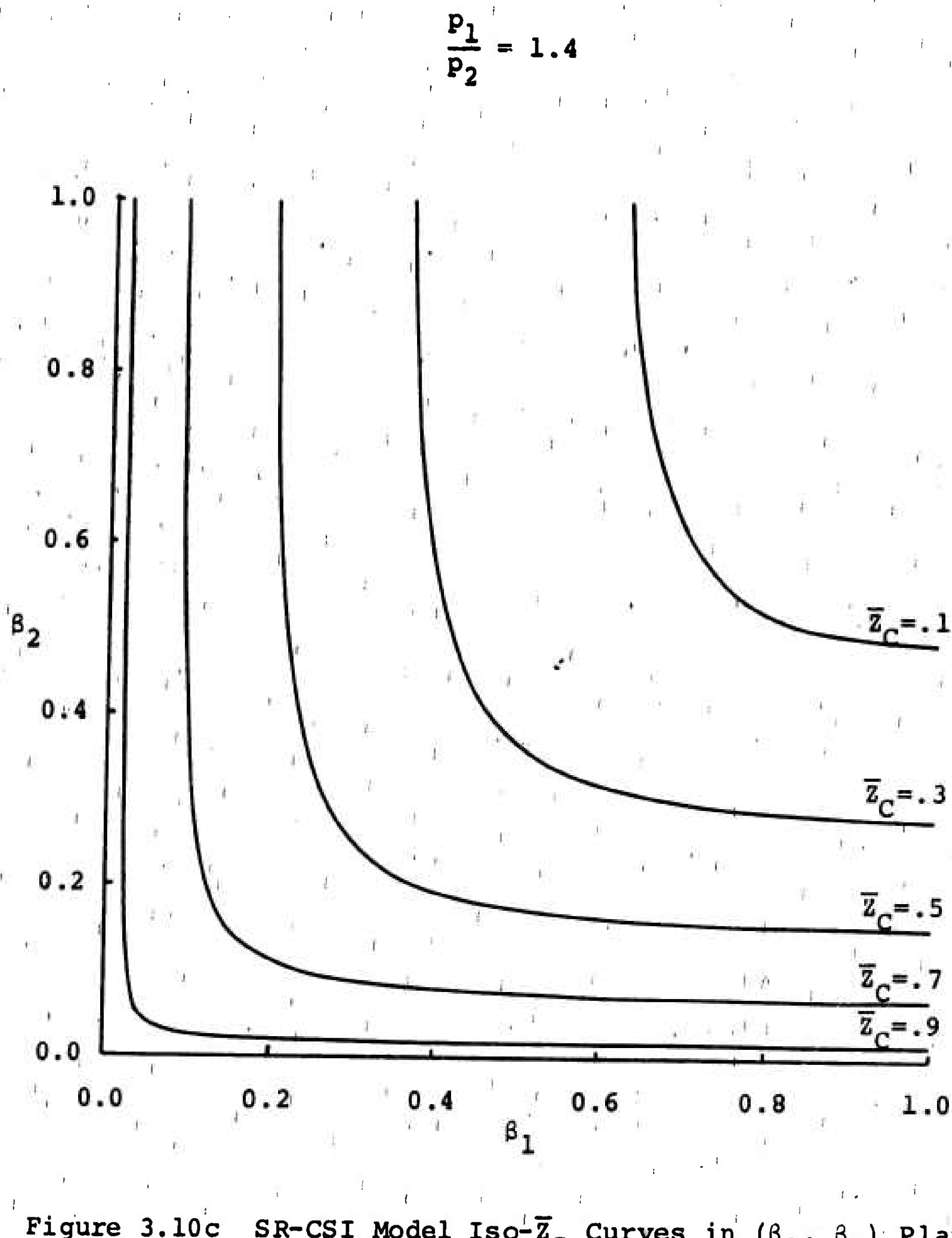

Figure 3.10c SR-CSI Model Iso-$\overline{Z}_C$ Curves in (β_1, β_2) Plane

Equations D.5a and D.5b in Appendix D can then be used to express the tradeoff as

$$d = \frac{(1+\beta)\overline{Z}_C}{p_2(1+\beta)+p_1(1-\beta)\left[\frac{p_1}{p_2}\right]^{\frac{\beta}{1-\beta}}} ,$$

where $p_1 \leq p_2$. Figure 3.11 shows the SR-CSI $\overline{Z}_C = .5$ curves in the (β,d) plane for several values of p_1. The coordinates of points on other iso-$\overline{Z}_C$ curves can be found from these curves in the same manner as for the SR-LSI iso-$\overline{Z}_L$ curves. These curves are quite similar to the corresponding SR-LSI $\overline{Z}_L = .5$ curves. The two families of curves have the same end points. But the CSI curves are slightly more gently curved. This reflects the inherent marginal value of having the contact times available in making the target location guess decision.

To investigate the robustness of the SR-CSI expected utility, a computer program was developed to examine the sensitivity of the SR-CSI expected utility to the allocation along $T_1 + T_2 = T$. Figures 3.12a to 3.12c show normalized CSI expected utility values as functions of T_1 along with the corresponding expected utility functions shown in Figures 3.6a to 3.6c. The sets of curves are somewhat similar, but the CSI version curves show less

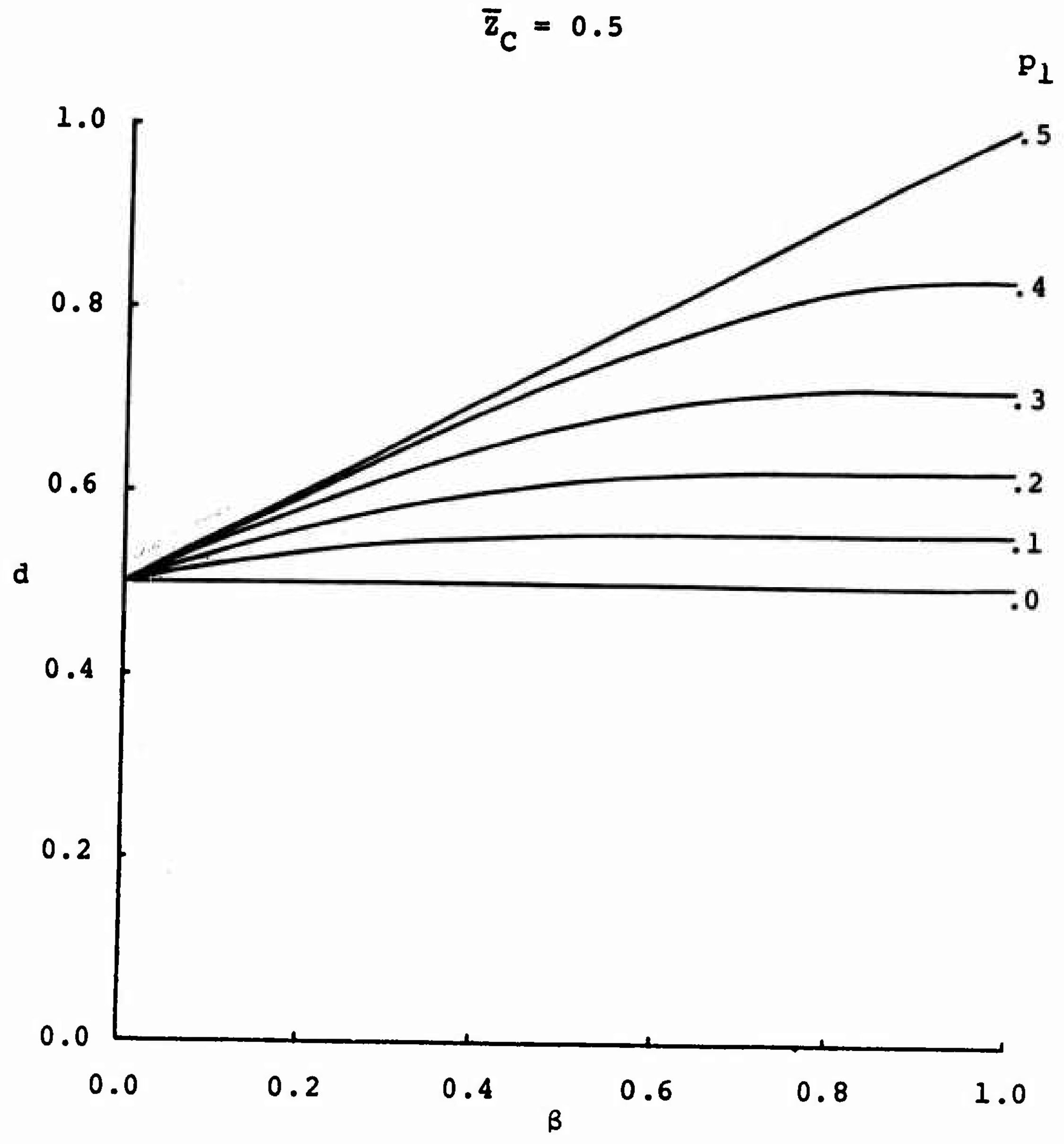

Figure 3.11 SR-CSI Model Iso-$\overline{Z}_C$ Curves in (β, d) Plane

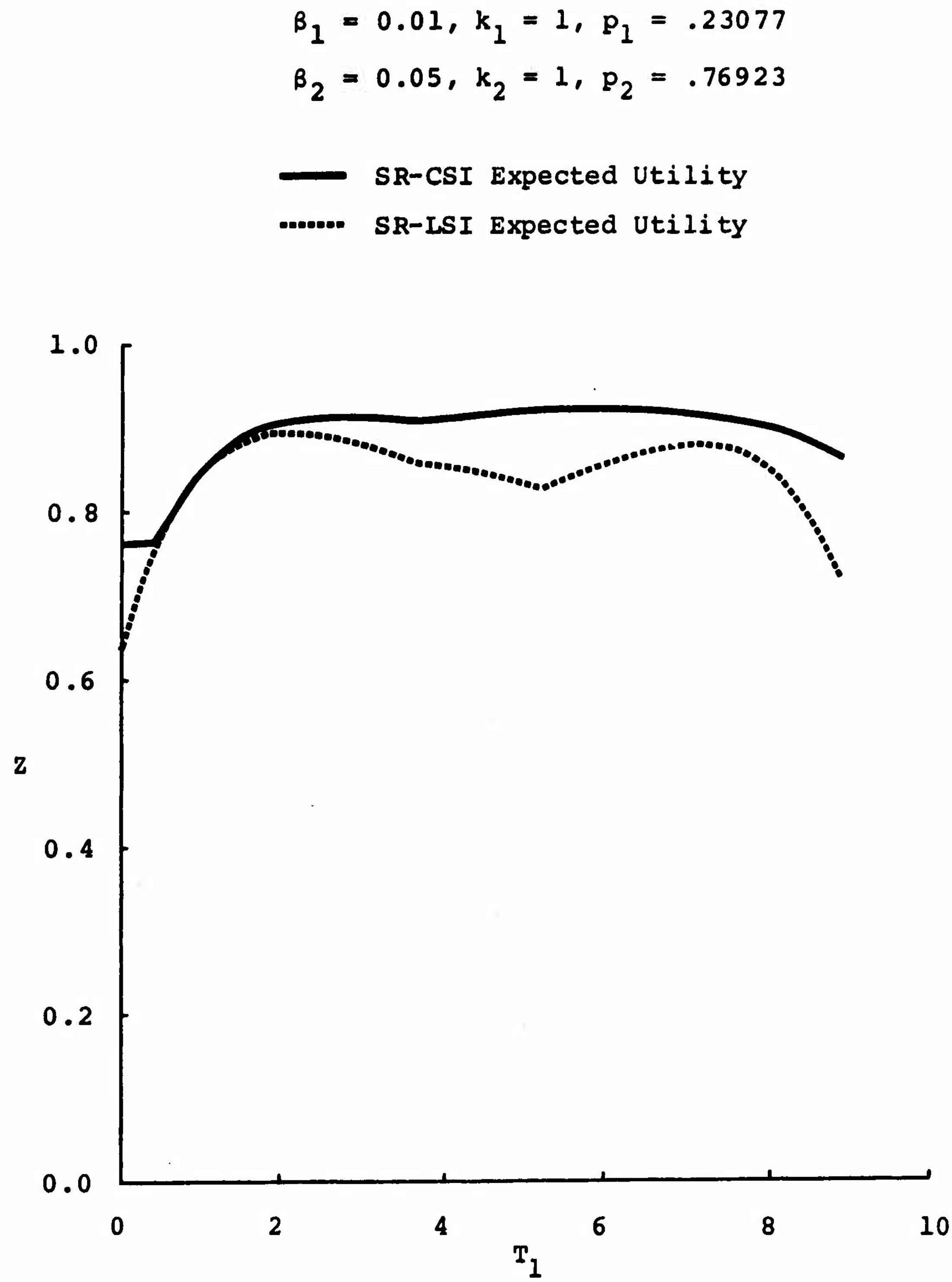

Figure 3.12a Normalized SR-CSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

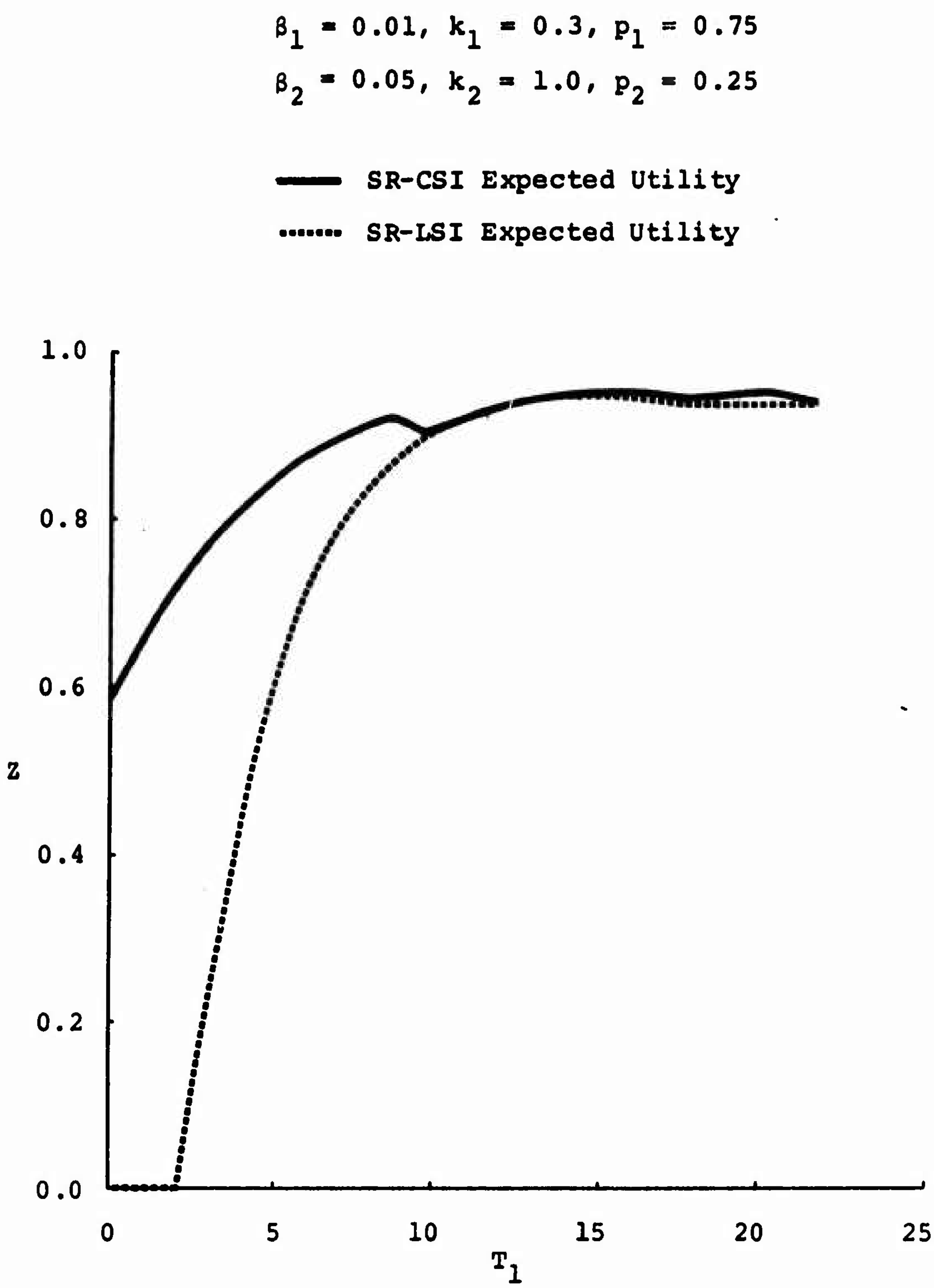

Figure 3.12b Normalized SR-CSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = T$

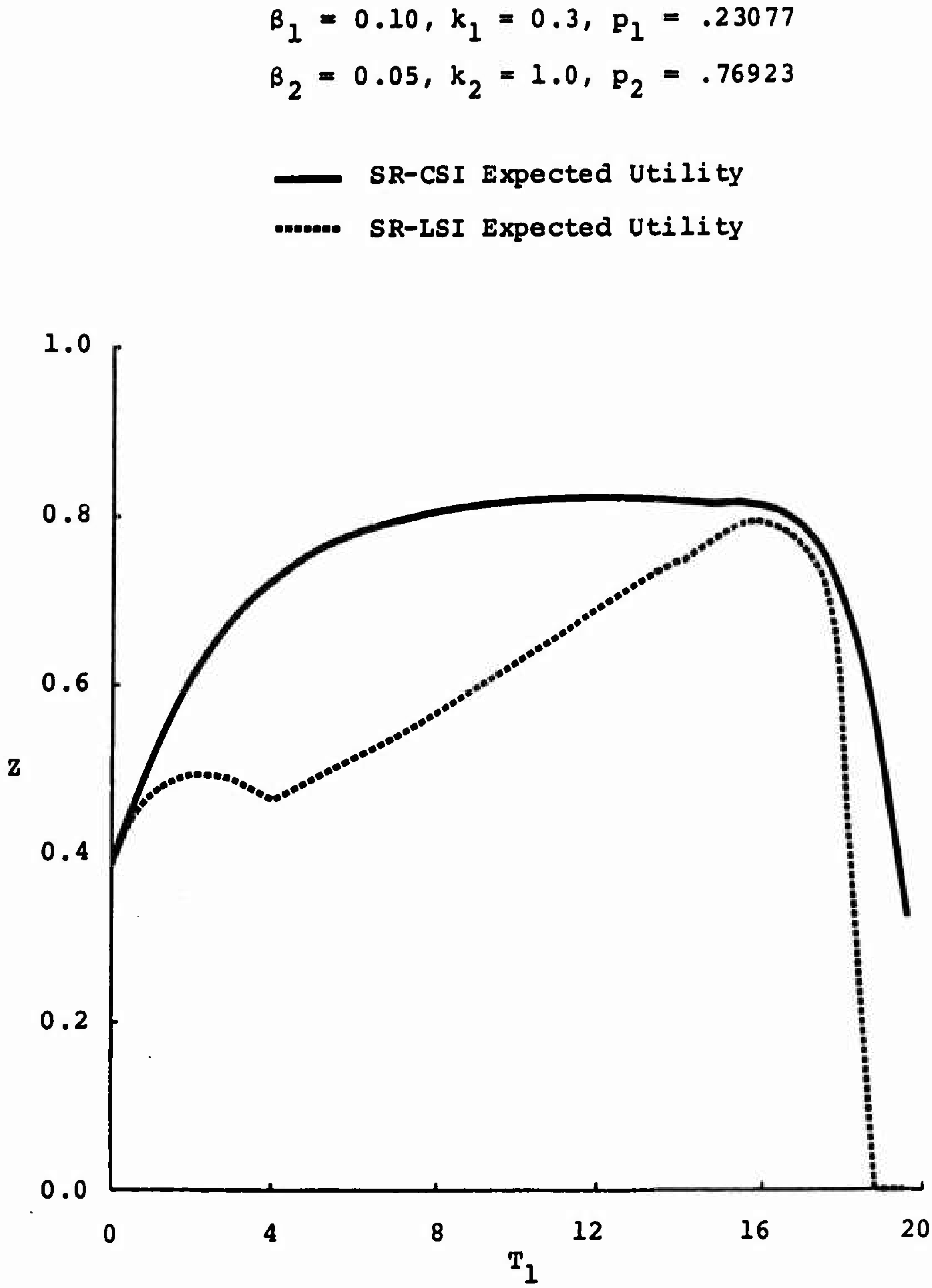

Figure 3.12c Normalized SR-CSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = T$

sensitivity to allocations than do the LSI curves. Thus, the use of the more sophisticated CSI model eliminates some of the sensitivity of expected utility to the allocation. That is, for large T the importance of allocating the search resources nearly optimally is greater for the LSI version than for the CSI version. Figure 3.13 shows the sensitivity of the SR-CSI expected utility to the allocations for series of values of T.

3.2.3 Summary of SR-CSI Results and Implications

The similarity of the CSI and LSI versions of the SR model leads to many similarities in results for these models. Therefore, most of the comments of section 3.1.3 apply to the CSI version as well as the LSI version. In this section we discuss the important qualitative differences between the results for these two models.

Optimal SR-CSI allocations follow optimal trajectories which coincide with the corresponding SR-LSI optimal trajectories for small and moderate values of T. But, for large T the CSI version optimal trajectories are unbounded lying along straight lines of positive slope in the (T_1,T_2) plane. Also, the CSI optimal trajectories lack the final switch between conditional trajectories which usually characterize the LSI optimal trajectories.

The optimal SR-CSI expected utility values are identical to the corresponding SR-LSI expected utility values

$\beta_1 = 0.01,\ k_1 = 1,\ p_1 = .23077$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .76923$

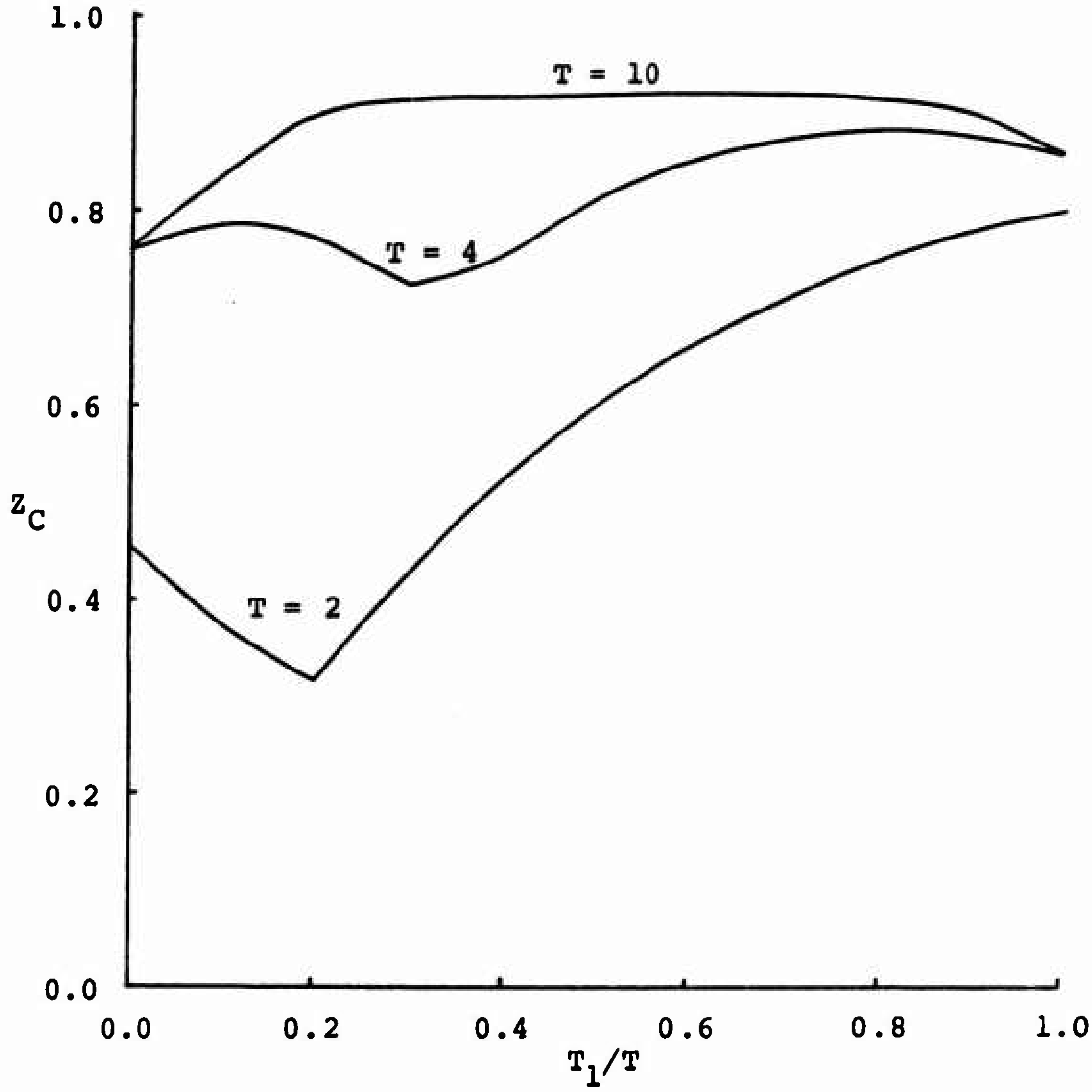

Figure 3.13a Normalized SR-CSI Model Expected Utility Sensitivity Dependence on Available Search Time

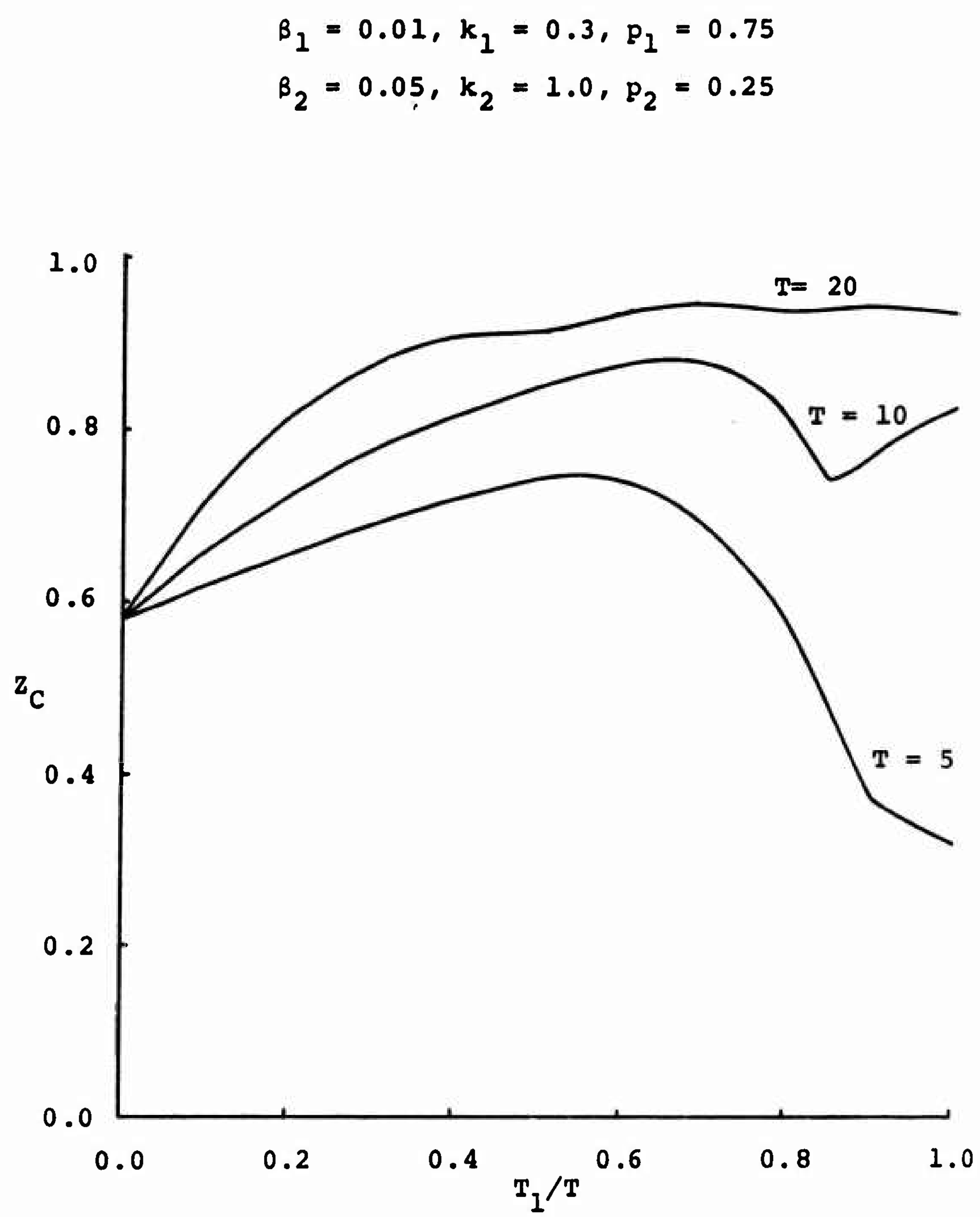

Figure 3.13b Normalized SR-CSI Model Expected Utility Sensitivity Dependence on Available Search Time

for small and moderate values of T; for large T the optimal SR-CSI expected utilities are slightly larger. The important difference between the SR-CSI and SR-LSI expected utility functions is in their sensitivity to changes in the allocation, T_1 and T_2, for large T. For the LSI version the expected utility decreases as a function of T_1 (or T_2 or T_1 and T_2 along a line of positive slope) if the amount of time allocated is large. But, the CSI expected utility function is non-decreasing in T_1 and T_2 for all possible allocations. For the CSI version any possible allocation of large T with $T_1 >> 0$ and $T_2 >> 0$ will yield an expected utility which is roughly the same magnitude as the optimal expected utility. Thus, if the contact times are used in making the target location guess decision, the expected utility loss for a non-optimal search allocation is less for large T than if only the numbers of contacts in the two boxes are used for making the target location guess.

3.3 Adaptive Complete Search Information (ACSI) SR Model

For both the LSI and CSI versions of the SR model the available time for searching in each box is fixed, independent of the search outcome. Thus if a contact occurs in one of the boxes before the planned search of the box is finished, the unused portion of the planned

search is not reallocated to the other box. Rather, this unused available searching time is wasted. For the adaptive complete search information version of the SR model let us assume that when the first contact is made this unused searching time is reallocated to the other box. In Appendix D the SR-CSI search model is modified to include such adaptive reallocation of the search resource.

The analysis of this AR-ACSI model is very similar to that for the unadaptive SR-CSI model. The only difference in the definition of the problem comes in the probability distribution of $\underline{t}$. Equation 25 becomes

$$\overline{F}(\underline{t}) = \begin{cases} p_1 e^{-k_1 t_1 - \beta_2 k_2 t_2} + p_2 e^{-\beta_1 k_1 t_1 - k_2 t_2} & \text{for } t_1 + t_2 < T \\ 0 & \text{for } t_1 + t_2 \geq T. \end{cases}$$

And equation 26 becomes

$$\underset{\underline{T}}{\text{Maximize}} \quad Z = \iint\limits_{0 \leq t_1 + t_2 \leq T} \max_{i=1,2} \{p_i^!(\underline{t}, \underline{T})\} d\overline{F}(\underline{t})$$

$$\text{Subject to} \quad T_1, T_2 \geq 0, \quad T_1 + T_2 \leq T.$$

The two conditional trajectories become the positive portions of the lines b and c together with the segments of the T_1 and T_2 axes connecting these two lines to the origin as shown in Figure 3.7. However, all points in the non-negative quadrant above line c along $T_1 + T_2 = T$

yield the same values of the SR-ACSI expected utility as the point on line c having $T_1+T_2 = T$. Similarly, all points in the non-negative quadrant below line b along $T_1+T_2 = T$ yield the same values of SR-ACSI expected utility as the point on line b having $T_1+T_2 = T$. Therefore, solutions to the ACSI version of the model are unique only for small T. In fact, for any T there exists an optimal allocation with all of the available search time allocated to one of the boxes.

3.3.1 Comparison of SR-ACSI Model with SR-CSI and SR-LSI Models

The computer program which computes approximate numerical solutions for the SR-CSI model also computes numerical solutions for the ACSI version. Figures 3.14a to 3.14d present the ACSI version optimal trajectories for the same cases depicted by Figures 3.1a and 3.1d for the LSI version. The SR-ACSI optimal trajectories consist of alternating segments of two conditional trajectories similar to those of the SR-CSI model. The two SR-ACSI conditional trajectories consist of the portions of lines b and c in the positive quadrant and the segments of the coordinate axes connecting these rays with the origin. The same number of switches between conditional trajectories was found for the SR-CSI and SR-ACSI model optimal trajectories for all cases examined. The same criteria hold

$\beta_1 = 0.1,\ k_1 = 1,\ p_1 = 0.8$

$\beta_2 = 0.3,\ k_2 = 1,\ p_2 = 0.2$

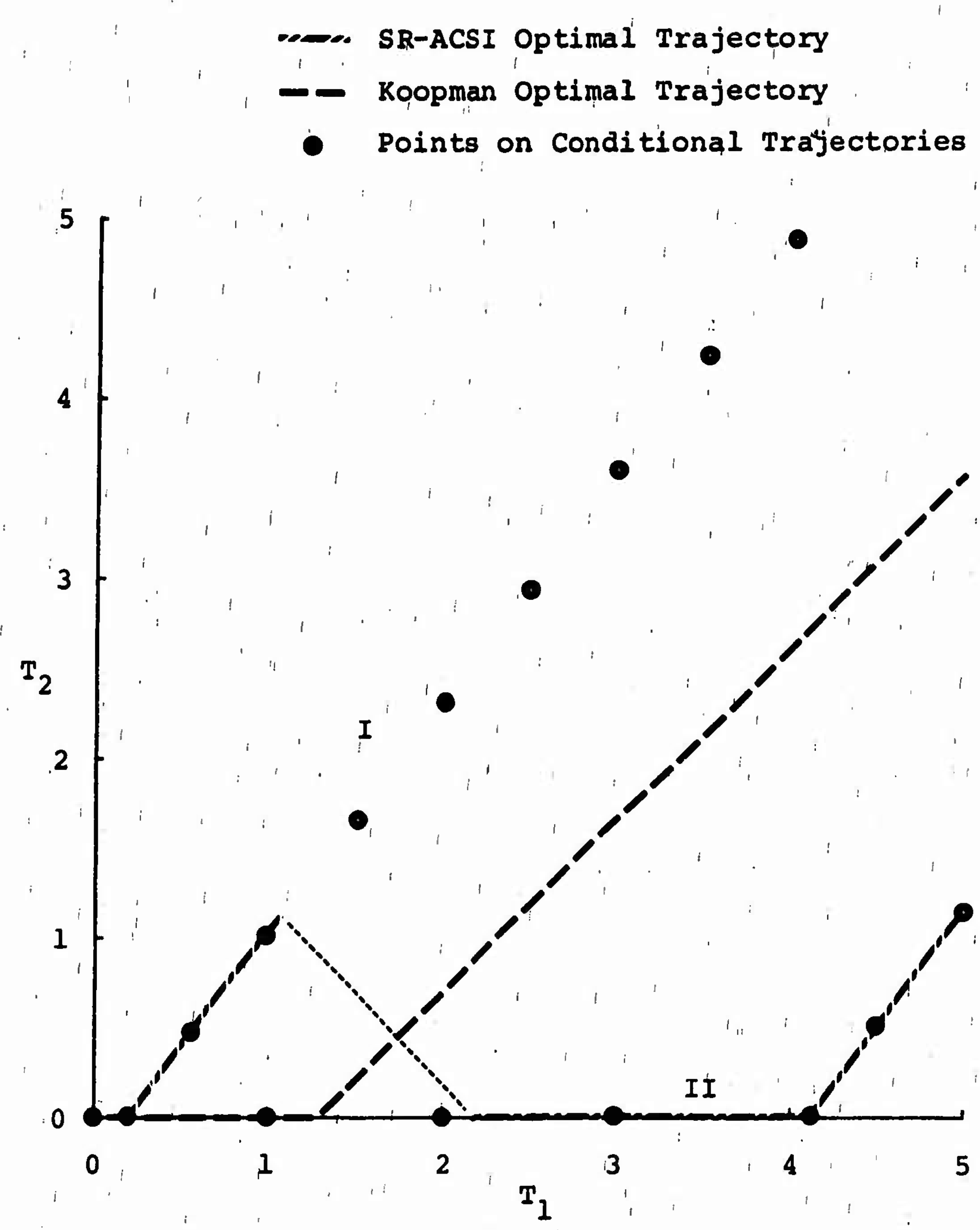

Figure 3.14a SR-ACSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.01$, $k_1 = 0.5$, $p_1 = .47368$

$\beta_2 = 0.10$, $k_2 = 1.0$, $p_2 = .52632$

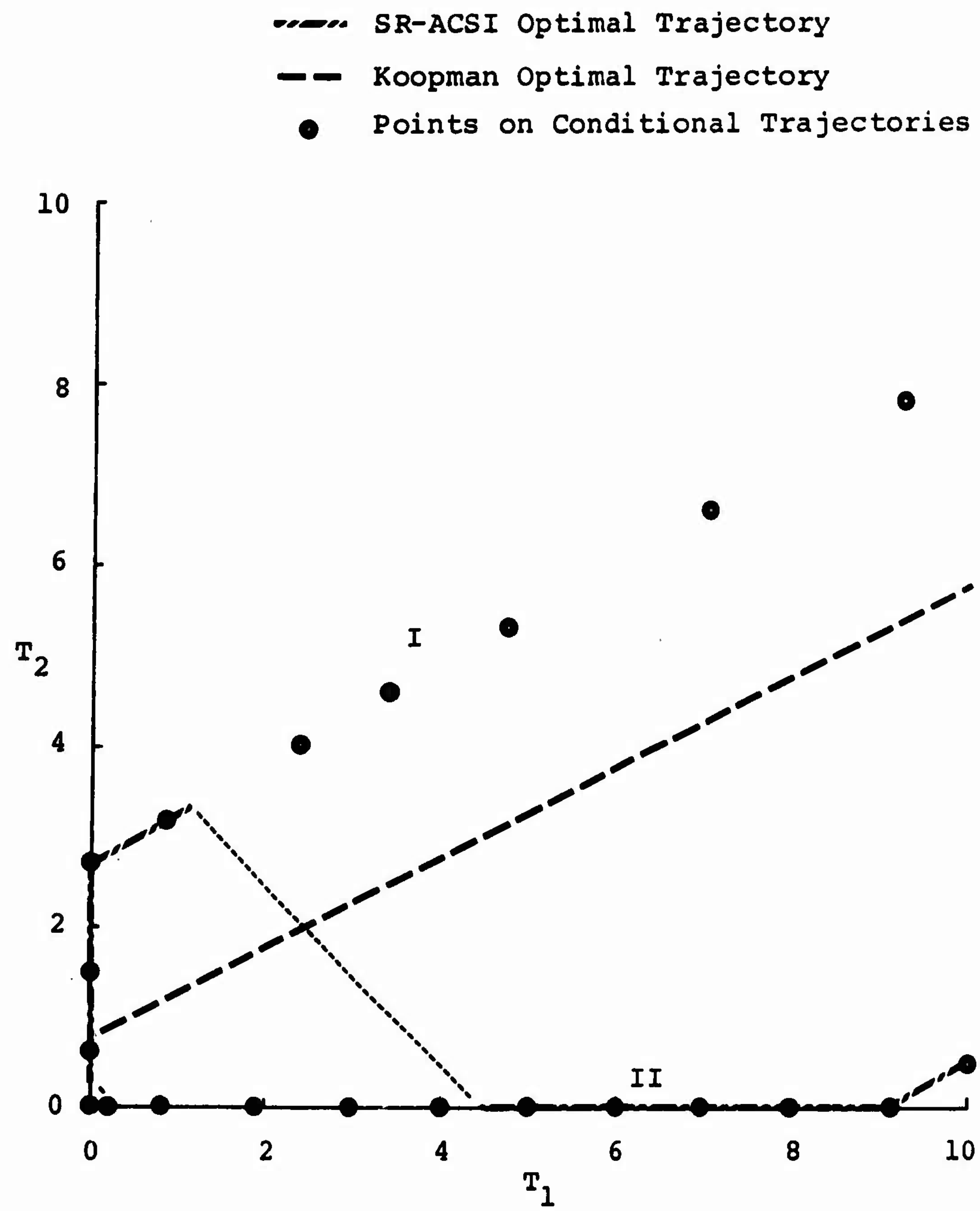

Figure 3.14b SR-ACSI Model and Koopman Optimal Trajectories

$\beta_1 = 0.10,\ k_1 = 1,\ p_1 = .47368$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .52632$

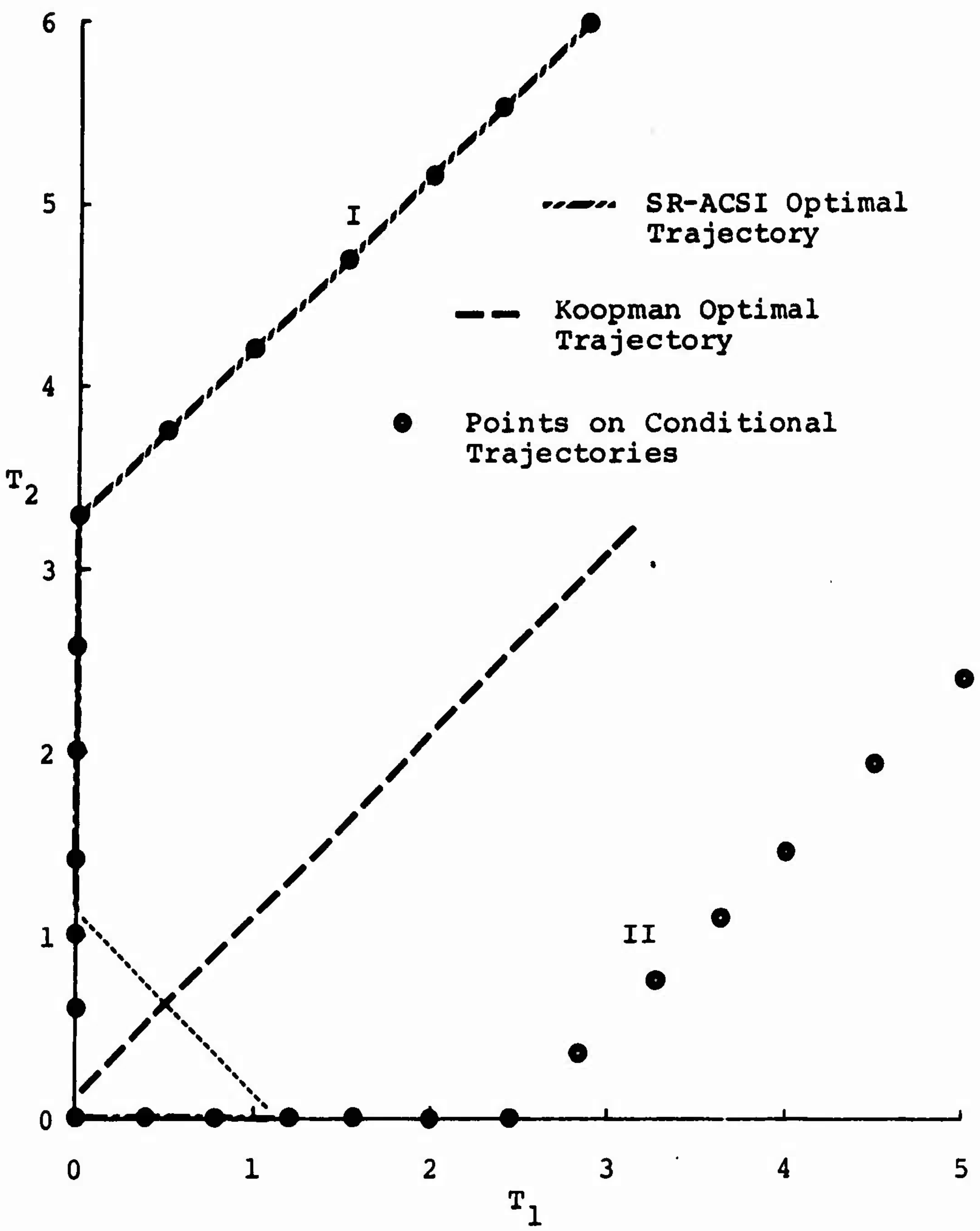

Figure 3.14c SR-ACSI Model and Koopman Optimal Trajectories

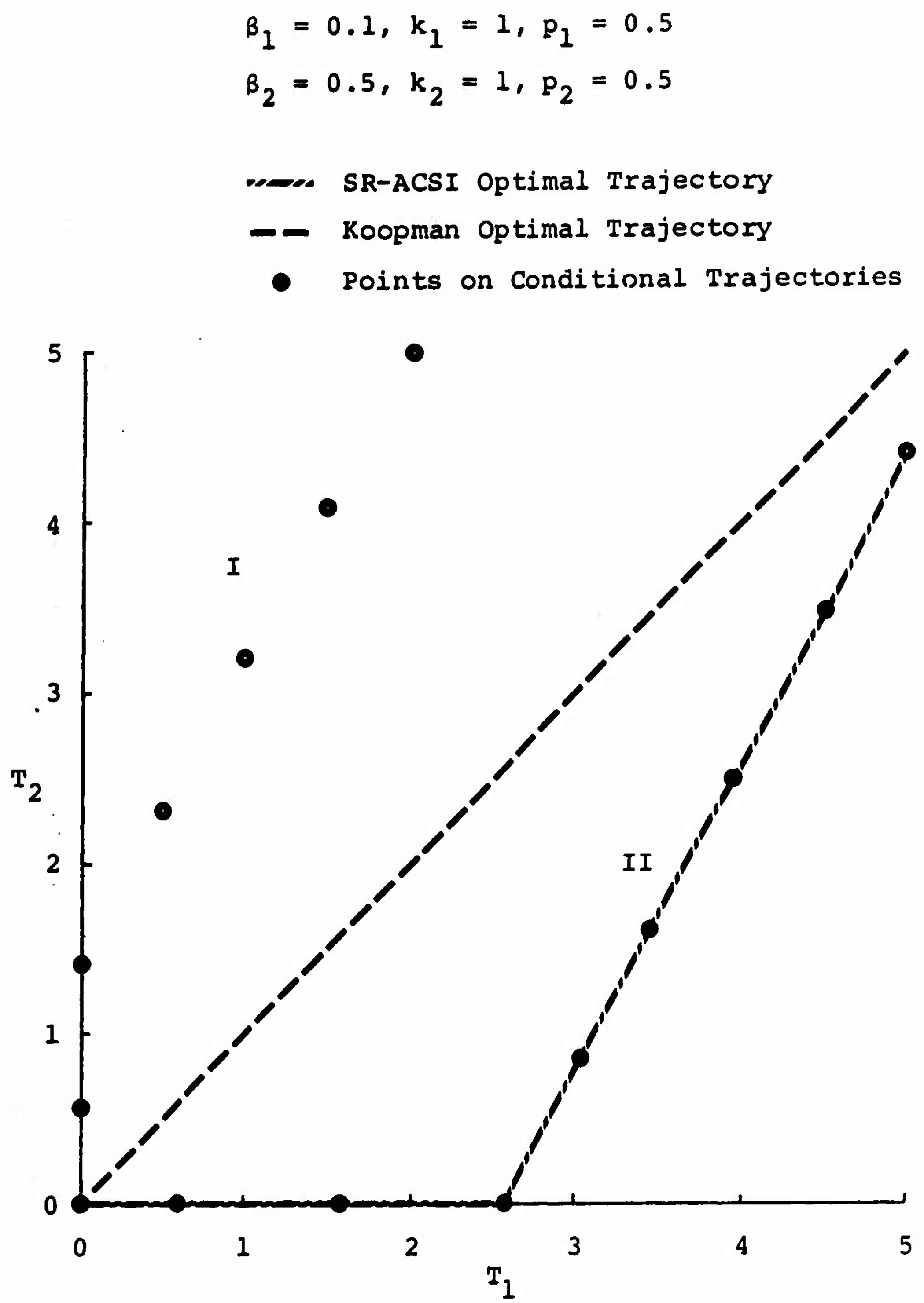

Figure 3.14d SR-ACSI Model and Koopman Optimal Trajectories

for the determining which conditional trajectory is optimal for the small, moderate and large ranges of T. But, the switch times for the SR-ACSI model were always found to be higher than the corresponding SR-CSI switch times. LSI conditional trajectories coincide with the corresponding CSI conditional trajectories for small T, depart from the CSI conditional trajectories for intermediate values of T and end on the corresponding ACSI conditional trajectories.

Figures 3.15a to 3.15d depict the SR-ACSI version optimal expected utility values, Z_A, as functions of T for the same cases depicted by Figures 3.2a to 3.2d for the LSI version and by Figures 3.9a to 3.9d for the CSI version. The ACSI curves differ from the CSI curves only in minor detail -- the contribution of the adaptive reallocation of search time as the search progresses to the optimal objective function is of minor value.

3.3.2 Sensitivity of SR-ACSI Results

<u>Behavior for Small T</u>

The optimal SR-ACSI allocations for small T are identical to those for the LSI and CSI versions of the model. And the optimal ACSI expected utility functions are only infinitessimally higher for small T than for these two more primitive related models. Therefore, the sensitivity

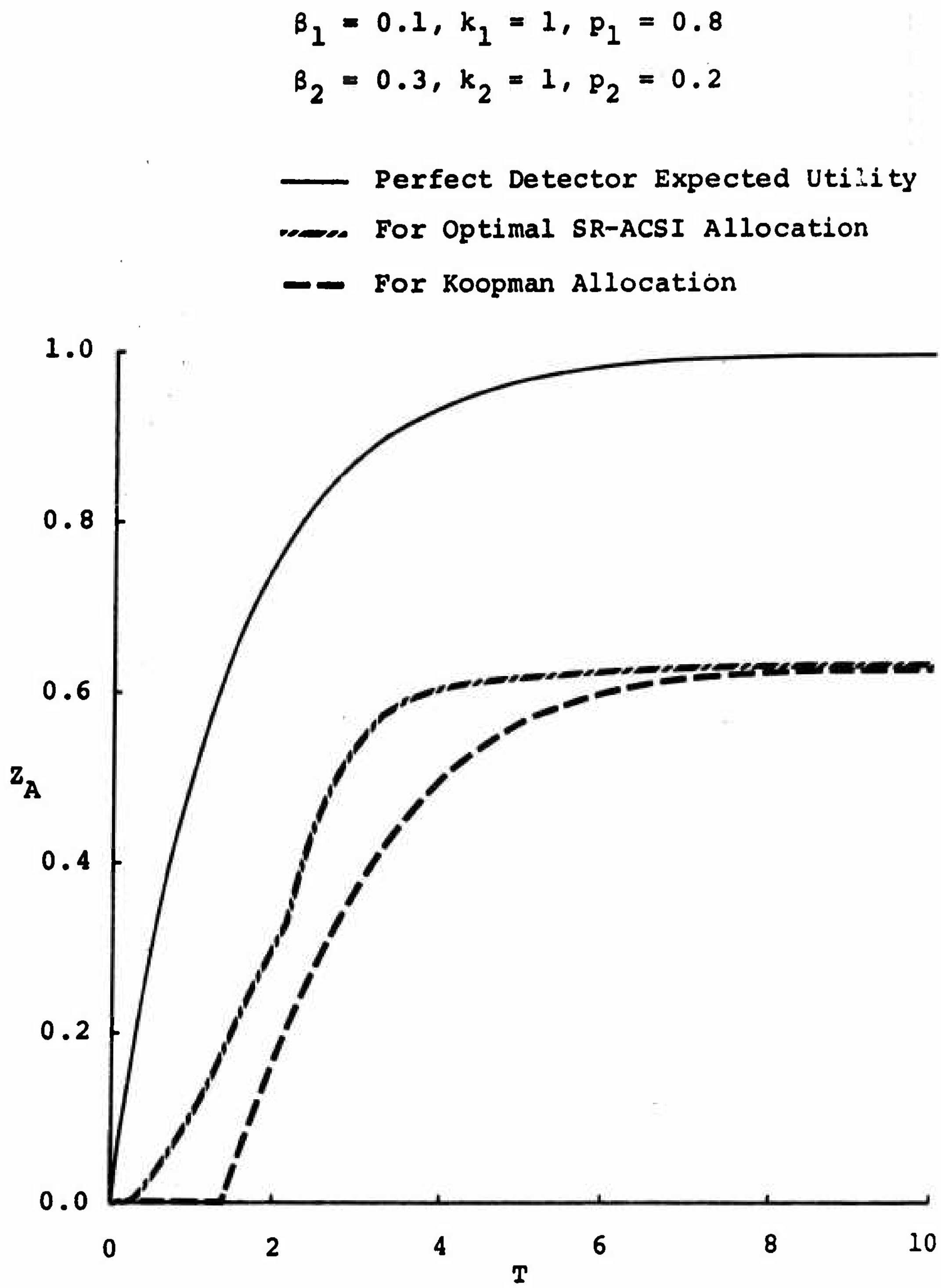

Figure 3.15a Normalized SR-ACSI Model Expected Utility vs Available Search Time

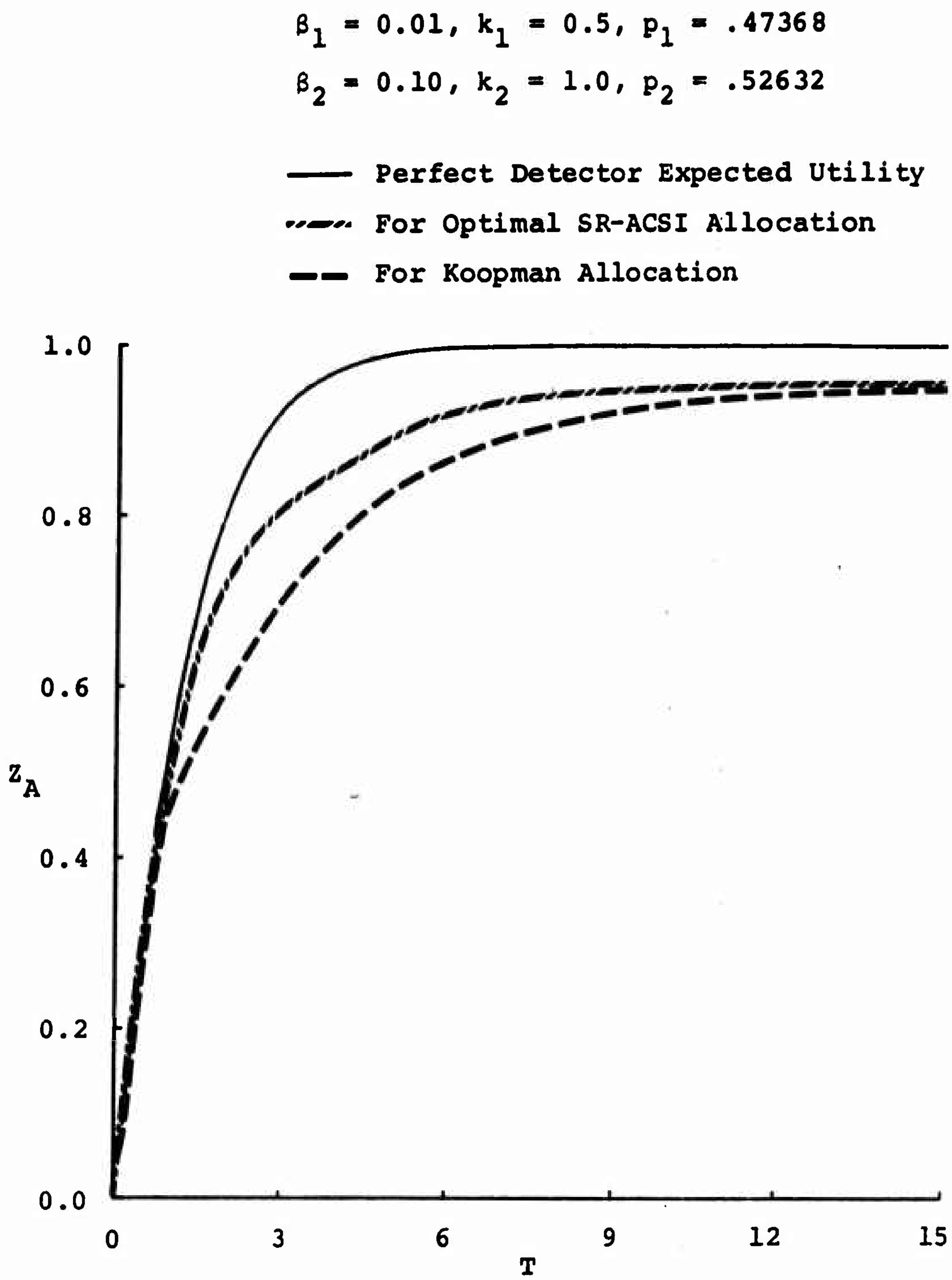

Figure 3.15b Normalized SR-ACSI Model Expected Utility vs Available Search Time

$\beta_1 = 0.10,\ k_1 = 1,\ p_1 = .47368$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .52632$

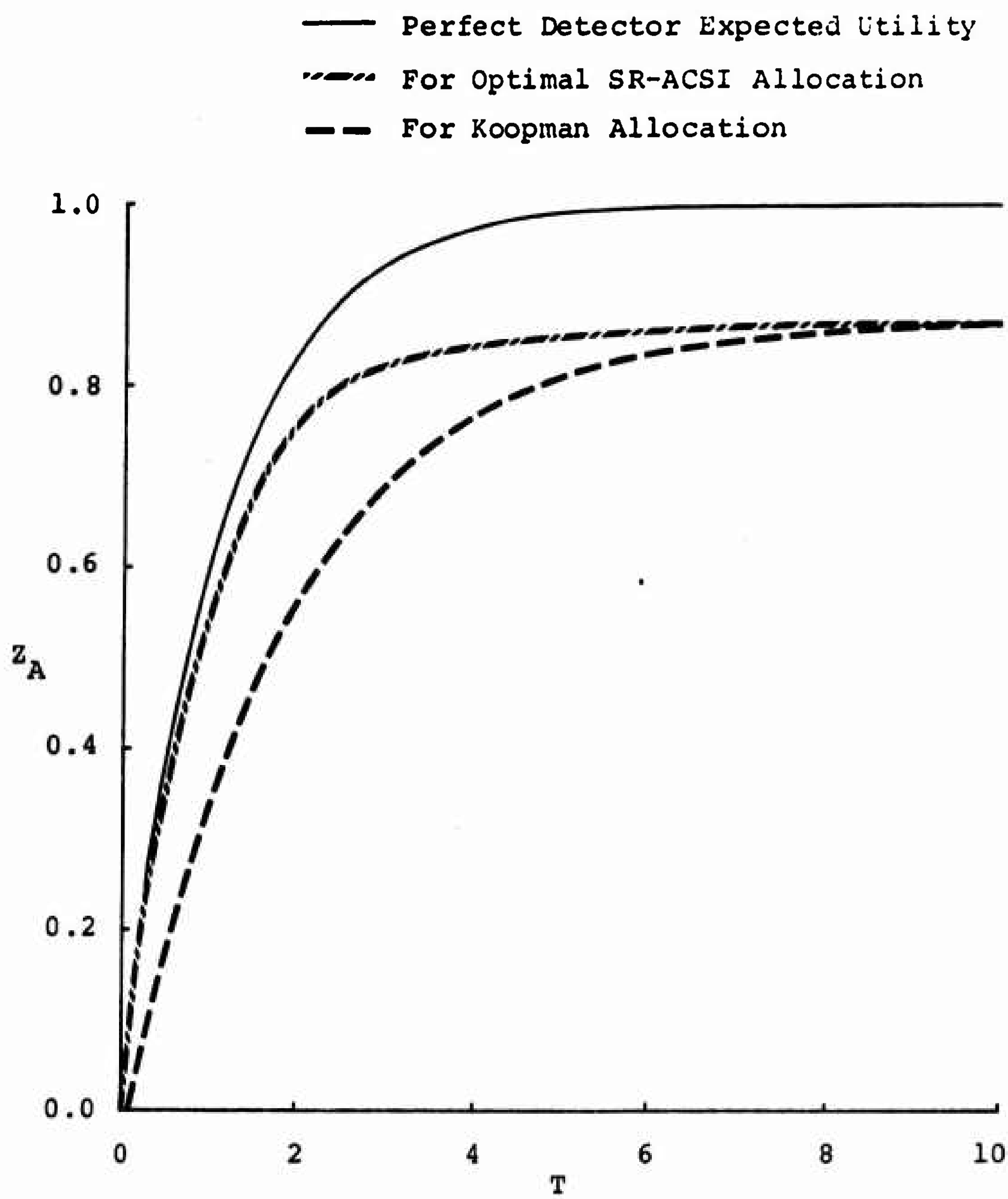

Figure 3.15c Normalized SR-ACSI Model Expected Utility vs Available Search Time

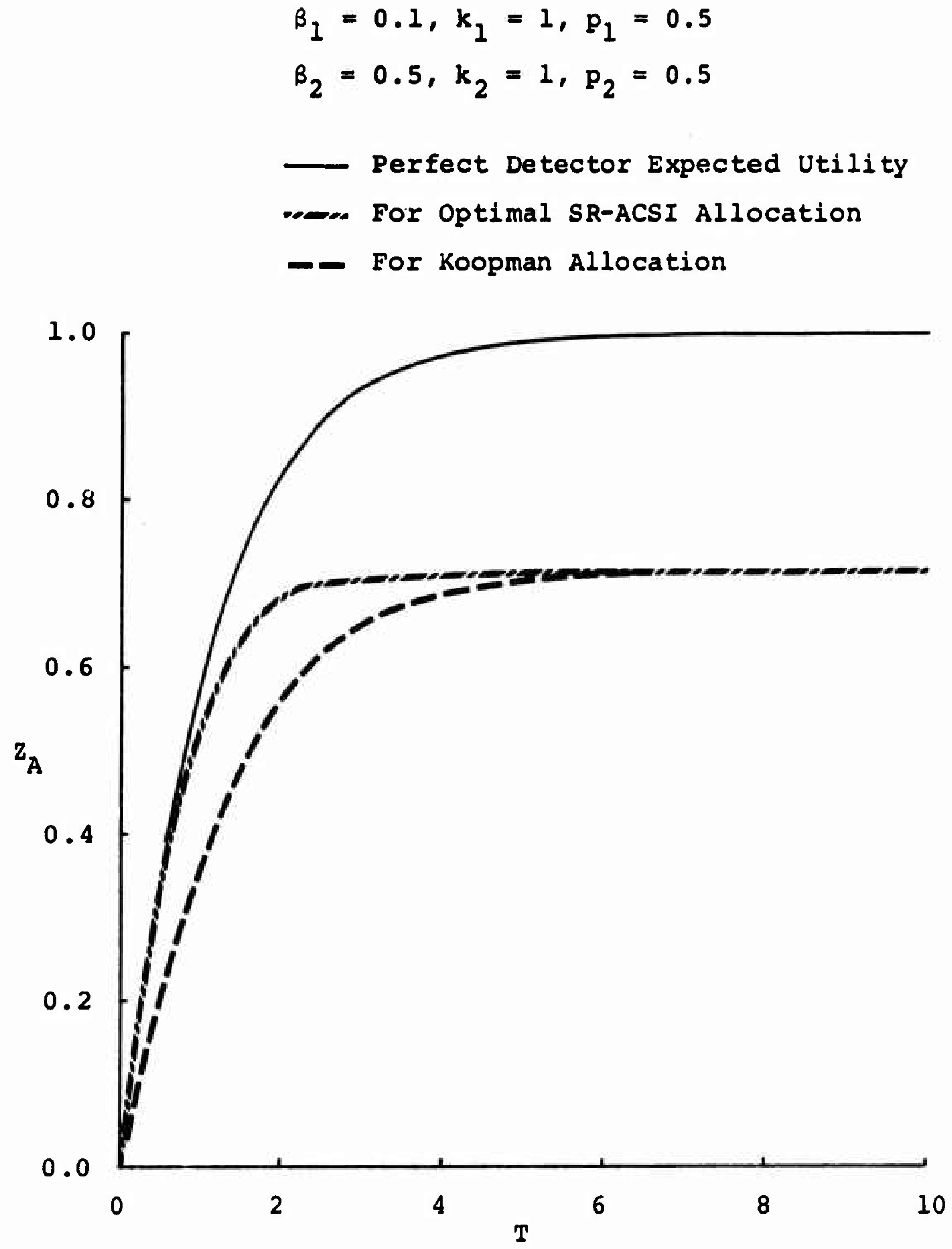

Figure 3.15d Normalized SR-ACSI Model Expected Utility vs Available Search Time

results derived for the LSI version hold for the ACSI version also.

Behavior for Large T

The ACSI and CSI versions of the SR model yield identical optimal expected utility values in the limit as $T \to \infty$. Therefore, the iso-$\overline{Z}_C$ curves computed for the CSI version (Figures 3.10 and 3.11) also hold for the ACSI version.

The computer program which computes the sensitivity of the CSI expected utility to the allocation along $T_1+T_2 = T$ also computes the ACSI version expected utility. Figures 3.16a to 3.16e show the normalized ACSI expected utility values as well as the CSI and LSI version expected utility values as functions of T_1 for the cases depicted in Figures 3.6, 3.12 and 3.13. These curves show that the objective function for the ACSI version is much less sensitive to the allocation than are objective functions of the LSI and CSI versions. Thus, for large T the chief benefit to be gained from using the ACSI model is that the expected utility is less sensitive to the allocation than is the case for the simpler LSI and CSI models.

3.3.3 *Summary of SR-ACSI Results and Implications*

Optimal SR-ACSI allocations can be described with the use of optimal trajectories which are very similar in

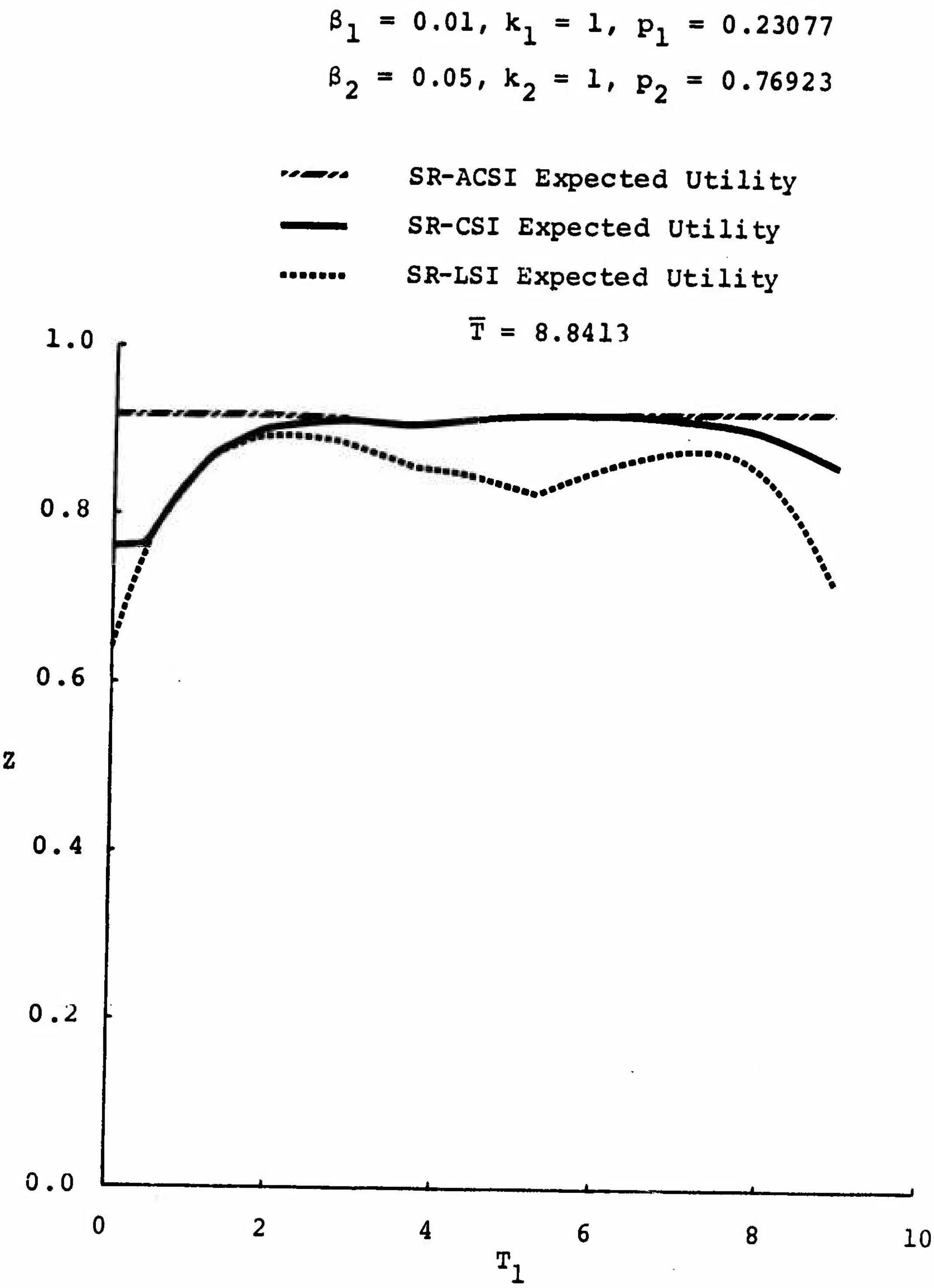

Figure 3.16a Normalized SR-ACSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \overline{T}$

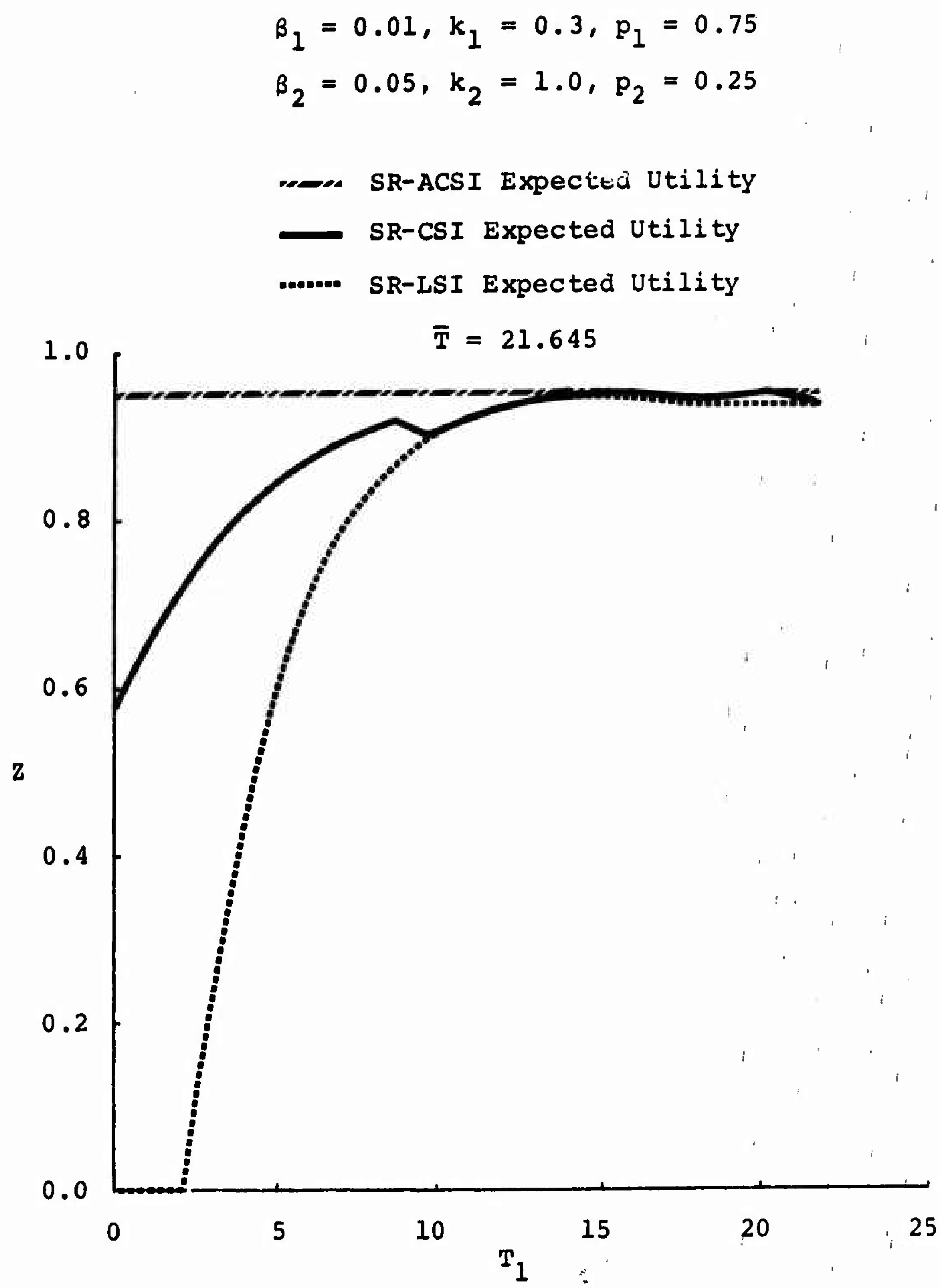

Figure 3.16b Normalized SR-ACSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

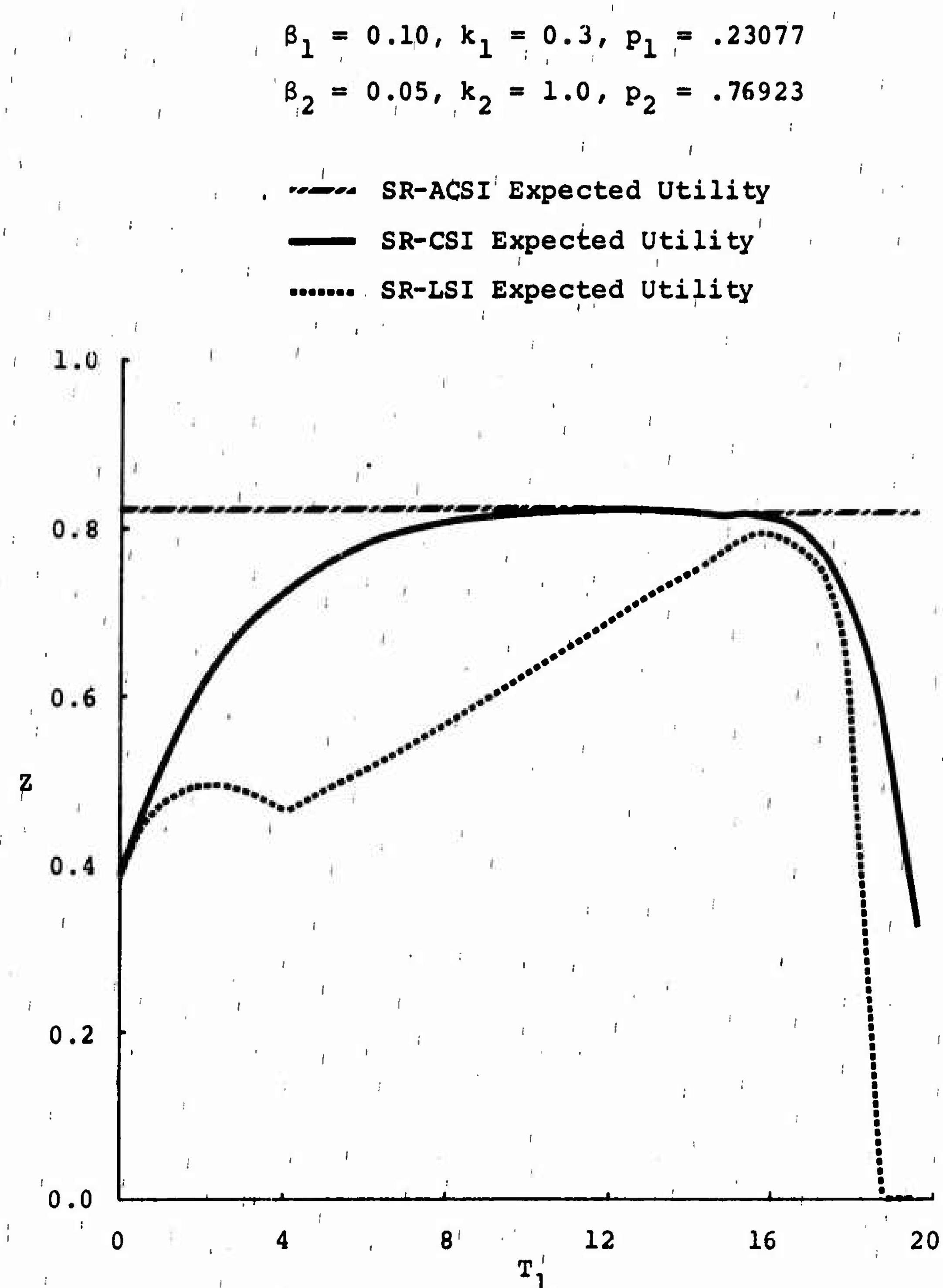

Figure 3.16c Normalized SR-ACSI Model Expected Utility Sensitivity to Allocation at $T_1 + T_2 = \bar{T}$

$\beta_1 = 0.01,\ k_1 = 1,\ p_1 = .23077$

$\beta_2 = 0.05,\ k_2 = 1,\ p_2 = .76923$

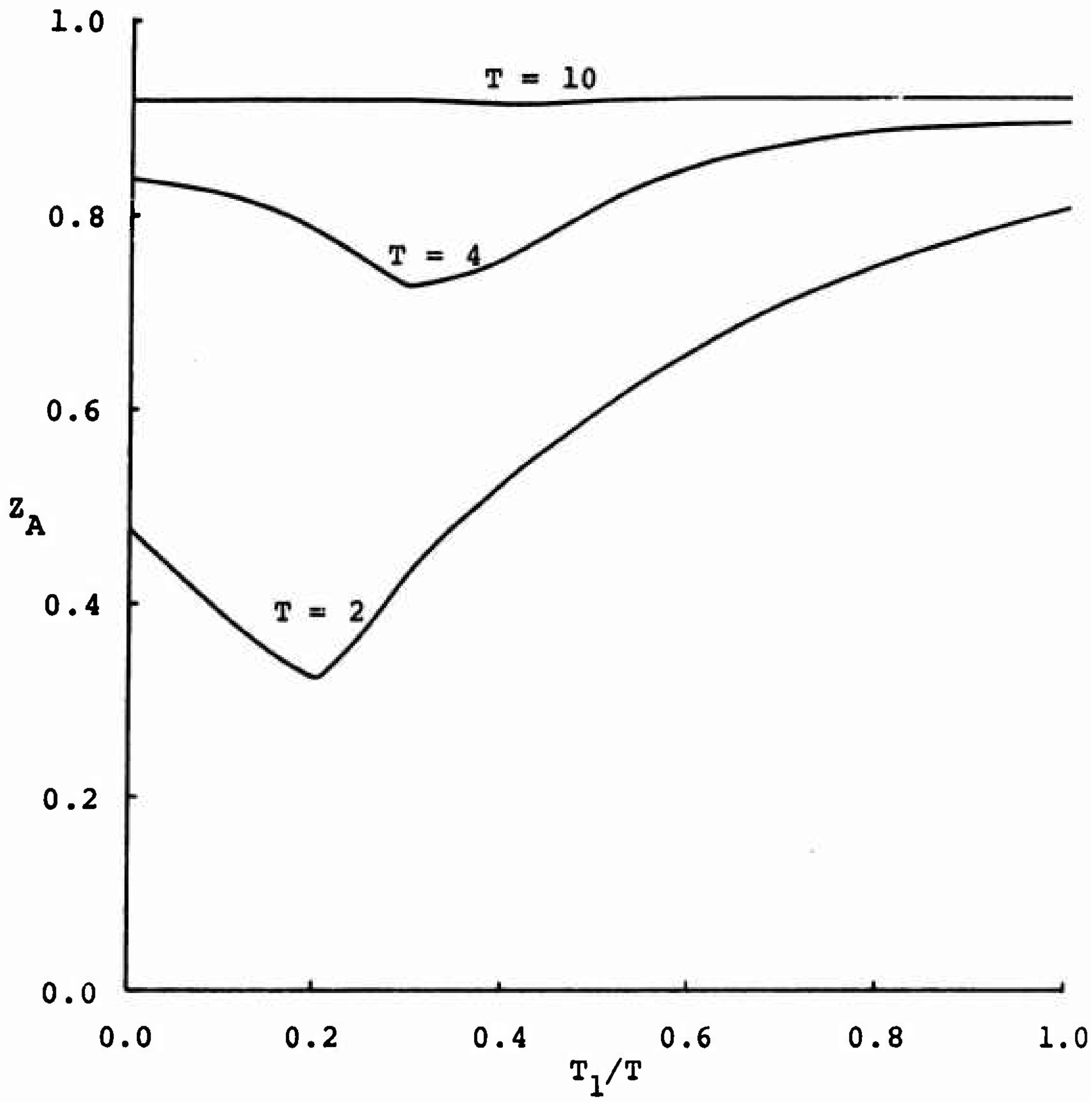

Figure 3.16d Normalized SR-ACSI Model Expected Utility Sensitivity Dependence on Available Search Time

$\beta_1 = 0.01, k_1 = 0.3, p_1 = 0.75$

$\beta_2 = 0.05, k_2 = 1.0, p_2 = 0.25$

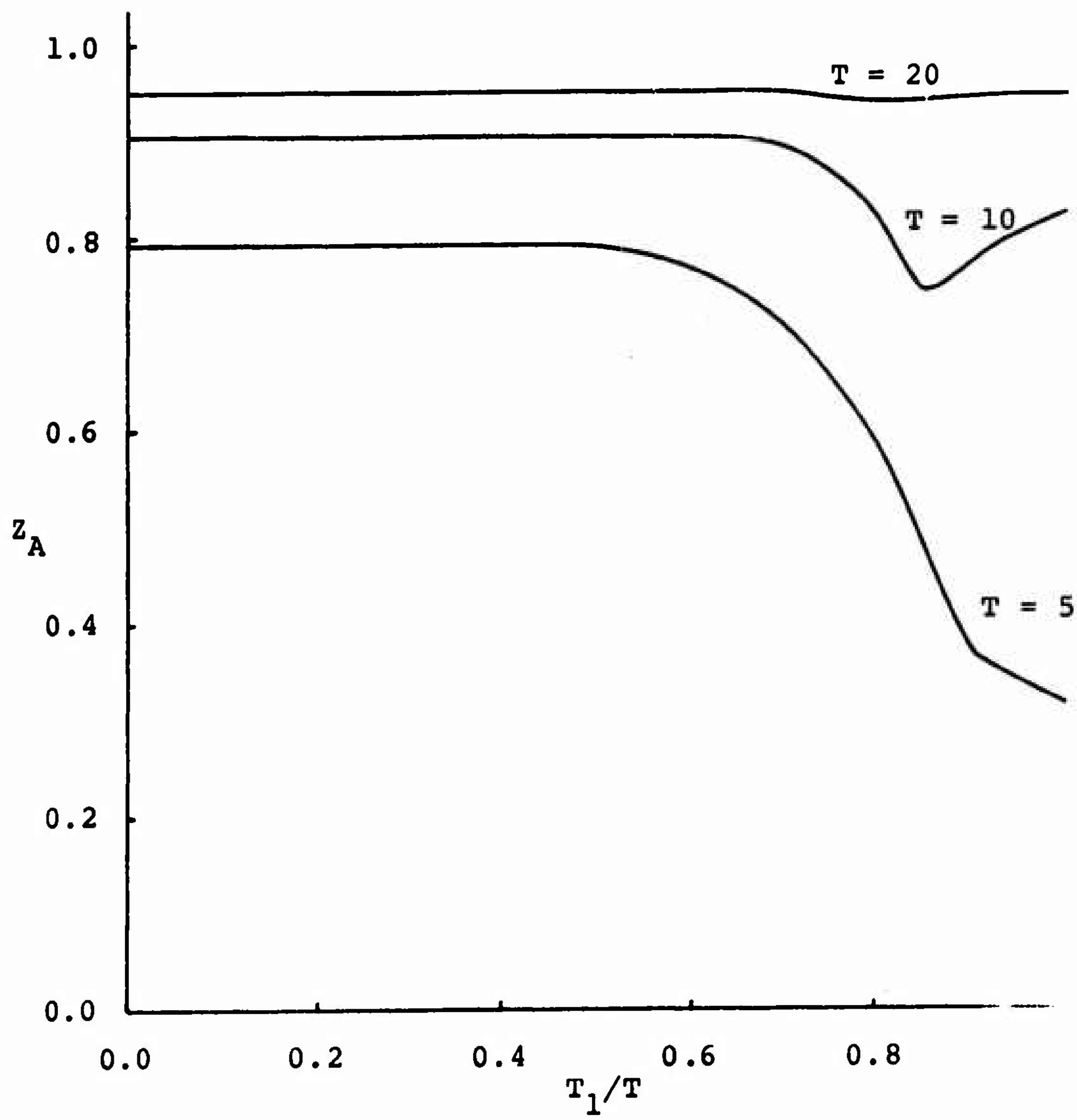

Figure 3.16e Normalized SR-ACSI Model Expected Utili... Sensitivity Dependence on Available Sea... Time

general appearance to the SR-CSI optimal trajectories. But, the relationship between optimal trajectories and the optimal allocation patterns for the SR-ACSI model are different from the corresponding SR-CSI relationship. In the SR-CSI model, only points on the optimal trajectory can be optimal. But, in the SR-ACSI model, for any given value of T all feasible points along $T_1+T_2 = T$ on one side of the optimal trajectory are also optimal. Thus, for any T there exists an optimal search plan which concentrates all search effort in one of the two boxes until either a contact is made or the available search time is exhausted. The box in which this optimal one box search begins depends on the amount of available search time, T, as well as the model parameters. For small T the optimal search begins in the box having the lower prior target location probability. For intermediate values of T the optimal one box search allocation is usually[1] to the box having the higher contact rate. For large values of T the optimal one box search allocation is usually[1] to the box having the lower false contact parameter.

For all T the optimal SR-ACSI expected utility values are either the same as or only slightly higher than the corresponding optimal SR-CSI expected utility values. There-

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[1]About 10% of the cases examined violate this rule.

fore, the ability to reallocate unused search time when the first contact is made contributes only marginally at best to the attainable expected utility.

CHAPTER 4

SUMMARY AND FUTURE RESEARCH

4.1 Summary of Results

4.1.1 UR Model

The distinction between favorable and unfavorable response process initiation with the corresponding false detection modeling affects the indicated search allocation by altering the expected utilities associated with unlimited searching in each of the two boxes. While these limiting expected utilities are necessarily positive for the Koopman model, either or both of the relevant limiting expected utilities can be negative for the UR model with false detections. Therefore, searching may be harmful in one or both boxes.

If searching is harmful in one or both boxes, the Koopman search allocation plan may lead to disasterous results as shown in Figure 2.2. However, if searching is useful (has positive expected utility) in both boxes, the Koopman search allocation yields expected utility values which approach the optimal values as the amount of available search time increases without bound. For cases in which the false detection and response process parameters are the same for the two boxes, the Koopman search is optimal for any amount of available search time.

Searching in one of the boxes is harmful only in cases involving high potential risk (β_i high, f_i negative) and low potential gain (d_1 low). The Koopman search model is obviously inappropriate for such cases. For cases in which the Koopman model is reasonable, the Koopman allocations are close to the optimal UR allocations. The resulting expected utility loss from using the Koopman model rather than the UR model is moderate, approaching zero as the available search time grows large. Thus, the UR model can contribute significantly mainly in the analysis of search problems with low to moderate limits on the available search time.

4.1.2 SR Models

The optimal allocations for the SR search models are more complex than those for the Koopman and UR models. Even without false contacts ($\beta_1 = \beta_2 = 0$) the optimal trajectories may be non-increasing in T_1 and T_2. With false contacts present (β_1 and β_2 positive) the complexity of the optimal allocations increases. Multiple switching between two conditional trajectories becomes the usual optimal trajectory pattern. If $\beta_1 = \beta_2 = 0$, the ability to guess the target location eliminates the need to search in both boxes to optimize the SR objective. That is, guessing shifts the preferred allocation from a line of positive slope for the UR model to one of the axes. The

introduction of false contacts in the SR models moderates this shift. But, even with false detections present (β_i positive) there always exists a "one box" search (with T_1 or T_2 equal to zero) which attains a value of expected utility close to the optimal value. That is, for practical purposes it is never necessary to search in both boxes.

For the SR-ACSI model there exists an undesirable strip between lines b and c (given by (27) and (28)). A point within this strip can be optimal only if it is also on one of the axes. For sufficiently large T all feasible points on one side of the undesirable strip having $T_1 + T_2 = T$ are optimal solutions[1]. The location of the undesirable strip is determined by the prior target location probabilities, p_i, and the false contact parameters, β_i. The width and slope of the undesirable strip depend on the false contact parameters and the contact rates, k_i. Higher β_i values imply a narrower strip. For many moderate sets of parameter values, the optimal trajectory for the UR model lies within the undesirable strip.

The switching of the optimal SR trajectories from one side to the other of the undesirable regions usually follows the following pattern:

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[1]For the SR-LSI and SR-CSI models undesirable regions similar to the SR-ACSI undesirable strip exist. But for these models only points on the boundaries of the undesirable regions or on an axis can be optimal.

i. For small T the optimal allocation is concentrated in the box having the lower prior probability of containing the target.

ii. For moderate T the optimal allocation is usually along the conditional trajectory which is nearer to the axis corresponding to the higher total contact rate.

iii. For larger T the optimal allocation is usually along the conditional trajectory nearer to the axis corresponding to the lower false contact parameter, β_i.

iv. As T approaches $\overline{T}$ for the SR-LSI model, the optimal SR-LSI allocation usually lies along the conditional trajectory which is nearer to the axis corresponding to the higher false contact parameter.

That is, as T increases, the dominant role in determining which of the two conditional trajectories is optimal usually shifts from the prior probabilities to the total contact rates to the false contact parameters.

Because the optimal allocations, T_1^* and T_2^*, are not necessarily non-decreasing functions of T (amount of available search time), the value of T may be needed to compute the optimal search plan. Further, these optimal search plans are not related in any obvious way to optimal or desirable search plans for the corresponding problems with T unknown.

For the SR-LSI model the amount of available search time which is allocated to the two boxes is bounded (assuming that the false contact parameters are positive). But, the optimal allocations for the SR-CSI and SR-ACSI models are unbounded as T increases without bound. The SR-CSI optimal allocations for large T lies along straight lines of the same slope as the corresponding Koopman and UR allocations with the same true detection rates. Similar linearly increasing allocations of large T are also optimal for the SR-ACSI model. But, all allocations with $T_1+T_2 = T$ between the SR-ACSI optimal trajectory and one of the axes are also optimal. Thus, there always exists an optimal SR-ACSI allocation concentrated in one of the two boxes.

Only in rare cases is an optimal allocation approximately evenly shared. The corresponding parameter values for the two boxes are quite different for those cases in which approximately even allocations may be optimal. Thus, the search decision maker probably should avoid any impulse toward equal allocations to the two boxes in situations involving uncertainty.

The increased complexity of the optimal allocation patterns for the SR models compared to the Koopman and UR models leads to correspondingly more complex optimal expected utility structures. First, on an absolute scale

the optimal SR expected utility for $T = 0$ is positive (at least half of $\overline{Z}$) compared to zero for the UR model. While the optimal UR expected utility as a function of T is smooth and convex, the optimal SR expected utility may be neither smooth nor convex. The slope of this optimal SR expected utility function may be discontinuous having positive jumps at the points where the optimal trajectories switch between the two conditional trajectories.

The SR-LSI, SR-CSI and SR-ACSI version optimal expected utility functions of available search time are substantially identical for small and moderate amounts of available search time. For large (T_1+T_2) the SR-LSI optimal expected utility decreases asymptotically to the normalized value of zero as T increases without bound. In contrast, the SR-CSI and SR-ACSI optimal expected utility functions are strictly increasing functions approaching the same upper bound for large T.

The major difference in the SR-LSI, SR-CSI and SR-ACSI expected utility functions for small and moderate T is in their sensitivity. Particularly for moderate and large T the SR-ACSI expected utility functions show less sensitivity to T_1 with $T_2 = T-T_1$ than do the SR-CSI expected utilities. Similarly, the SR-CSI expected utilities exhibit less sensitivity to T_1 than do the SR-LSI expected utilities. The decreased sensitivity for the SR-CSI and

SR-ACSI expected utilities results from their ability to discriminate on the basis of when contacts occur between different search outcomes having the same numbers of contacts for the two boxes. The SR-ACSI decrease in expected utility sensitivity compared to that of SR-CSI model results from the increased flexibility of the ACSI search which adjusts the amounts of time allocated to the boxes whenever the first contact is made.

4.1.3 UR and SR Model Comparison

For small T the optimal UR (and Koopman) allocations are concentrated in the box having the higher $B_i k_i$ value, where $B_i = [(1-\beta_i)d_i+\beta_i f_i]p_i$. But, the optimal SR allocations begin in the box having the lower p_i value. The corresponding marginal expected utility functions at T = 0 for the UR and SR models are

$$\frac{dU}{dT} = [(1-\beta_1)d_1 + \beta_1 f_1]p_1 k_1 \quad ,$$

$$\frac{dZ}{dT} = (p_i-\beta_i p_j)k_i d \quad ,$$

where

$$[(1-\beta_1)d_1 + \beta_1 f_1]p_1 k_1 \geq [(1-\beta_2)d_2 + \beta_2 f_2],$$

$$p_i \leq p_j \; , \quad j = 3 - i .$$

Thus, $\frac{dU}{dT}$ is maximized by $p_1 = 1$ while $\frac{dZ}{dT}$ is maximized by $p_1 = p_2 = .5$. That is, the search contributions for these two types of models are greatest under circumstances which are exactly opposite in character. The UR search contribution is maximized by a state of perfect knowledge of the target location; the SR search contribution is maximized by the state of maximum uncertainty (entropy) regarding the target location. This difference reflects the different purposes for searching for these two types of models. In the SR models the purpose of the search is to discover or infer the target location. [illegible] in the UR model knowledge of the target location is insufficient to serve the decision maker's purposes. The searcher must *find* the target to obtain favorable response process results.

This difference in search purpose is accompanied by a fundamental difference in the nature of false contacts for the two types of models. For both types of models we define contacts as irreversible positive decisions that the target has apparently been found. Thus, contacts terminate the search in the box of contact for both types of models. For the UR model detections lead immediately to favorable response process initiation. False contacts are those detection decisions which lead to unfavorable response process initiation. In contrast, contact events in the SR models do not initiate the response process. The response process occurs only after the search has been

completed in both boxes. For the SR models any contact in the box containing the target tends to cause the decision maker to correctly guess the target location and, hence, obtain favorable response process results. That is, there are no false contacts associated with the box containing the target. Rather, the false contact phenomenon takes expression through the contact process in the box which does not contain the target. The SR false contact parameters are the ratios of intensity parameters for the conditional contact processes with and without the target present.

4.2 Future Research

The taxonomy of search problems given in Chapter 1 represents a conjectural view of what problem characteristics might be important to the analysis of search problems. The research has demonstrated that at least a small part of this conjectural view is productive in that consideration of response processes and false alarm phenomenon have marked effects on search strategies and returns. We have shown that these features can be implemented using rather straightforward mathematical techniques, and that the relationship between search and response processes is of central importance in the analysis of search problems with false contacts. These results suggest that research

should continue to explore the relationship between the search and response processes which will, of necessity, include the false alarm phenomenon. Other search and response features cited in the taxonomy should be investigated in conjunction with this direction. Sketches of three useful extensions of the research described in this report are given below.

The methods used in Chapter 3 can be generalized to deal with models having more than two boxes only at considerable computational expense. The sensitivity of our results to the two box assumption is of interest. Some insight into the character of corresponding multiple box model results can be found from sketching part of the solution to the special case with no false detections ($\beta_i = 0$). Corresponding to equation 18 we have

$$Z° = 1 + \max_{i=1,N} \left\{ p_i e^{-k_i T_i} \right\} - \sum_{j=1}^{N} p_j e^{-k_j T_j} \quad .$$

Therefore, the solution for any T is identical to the Koopman allocation confined to searching some subset of N-1 boxes. For small T this subset is all boxes except the one having the highest prior target location probability, p_i. For large T this subset is all boxes except

the one having the lowest detection rate[1], k_i. Thus, the switching phenomenon persists as N increases. If false detections are possible ($\beta_i > 0$), we expect the other qualitative characteristics of the solutions for the two box models to hold for the multiple box models as well. But, the development of multiple box and continuous search space models corresponding to those of this research remain as significant outstanding research topics.

In this research we assumed that the amount of available search time is known. This assumption seems particularly arbitrary resulting more from its convenience in the analysis than from an intuitive view of most real search problems[2]. Yet, this assumption was found to play an essential role in the SR model results. Corresponding models with some other treatment of the amount of available search time, T, are of interest. Note that if one has a probability distribution for T, the SR results of this research cannot, in general, be used to construct optimal search plans. Rather, one must express the expected utility as a function of the search plan and them maximize the resulting functional over the set of admissible

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[1]This follows from the fact that the rate parameter of the asymptotic exponential growth for large T is $1/\sum_{i \in S} 1/k_i$ where S is the subset of boxes searched.

[2]Inability of searchers to follow a prescribed strategy exactly is one effect which suggests viewing T as a random variable.

search plan functions which specify non-decreasing allocations to each box as a function of T. Such functional optimization problems are mathematically more difficult than the mathematical programming problems encountered in optimizing the same objective functions over the admissable allocations for a fixed T.

The detection processes assumed in this research are only the most rudimentary ones. The only output of the detection system is a single contact signal or contact event. This contact event, then, terminates the search. Further, the contacts, which are inherently interpretations of detector data, are made independent of the available target location probabilities. But, if possible, these contact decisions should be made on the basis of all the available information, including the initial target location probabilities. That is, response decisions should result from deliberate economic choices rather than from the arbitrary application of fixed rules for interpreting detector data independent of the operating environment. Models involving more explicit treatment of the detector data in the decision problem are needed to characterize such optimal economic search plans.

APPENDIX A

AN ALGORITHM FOR SEPARABLE, STRICTLY CONVEX PROGRAMMING PROBLEMS WITH ONE LINEAR BOUNDING CONSTRAINT

This appendix develops an algorithm for solving allocation problems like those of the UR search model problems of Chapter 2. The form presented here is a slight generalization of a similar algorithm by Charnes and Cooper (1958) and by Moore (1971). This algorithm can easily be modified for allocation vectors of arbitrary number of dimensions with substantially the same results. The assumptions regarding the characteristics of the objective function could be relaxed without altering the algorithm approach. However, to do so would clutter the development with superfluous details.

Let (x_1, x_2) be an allocation vector representing real, nonnegative quantities. Corresponding to each x_i let $F_i(x_i)$ represent a return function associated with x_i. Suppose F_i are differentiable with

i. $0 < F_i' < \infty$, (A.1a)

ii. $x < y \Rightarrow F_i'(x) > F_i'(y)$, and (A.1b)

iii. $\lim_{x \to \infty} F_i'(x) = 0$ (A.1c)

These conditions guarantee the existence of inverse functions, f_i, for each F_i'. Further, these inverse functions are continuous decreasing functions on intervals $(0, F_i'(0))$. Let us extend these inverse functions by defining

$$f_i(v) = \begin{cases} 0 & \text{for } F_i'(0) < v \\ F_i'^{-1}(v) & \text{for } v \leq F_i'(0). \end{cases} \tag{A.2}$$

<u>The Problem</u>

$$\text{Maximize } Z = F_1(x_1) + F_2(x_2) \tag{A.3}$$

$$\text{Subject to: } x_1, x_2 \geq 0, \; x_1 + x_2 \leq X > 0.$$

<u>The Solution</u>

The solution to the problem proceeds directly from the Kuhn-Tucker conditions which are:

$$F_i' \leq v, \tag{A.4a}$$

$$x_1(F_1'-v) + x_2(F_2'-v) = 0, \text{ and} \tag{A.4b}$$

$$v(x_1+x_2-X) = 0. \tag{A.4c}$$

Hadley (1964) shows that if (x_1^*, x_2^*), v^* satisfy these Kuhn-Tucker conditions and (x_1^*, x_2^*) is feasible for the problem, (x_1^*, x_2^*) solves the problem.

Since $F_i' > 0$, $v \leq 0$ cannot satisfy (A.4a). Therefore, we may assume that $v > 0$. Then (A.4c) reduces to

$$x_1 + x_2 = X. \tag{A.5}$$

Together, (A.4a) and (A.4b) imply that either

i. $x_i = 0$, or (A.6)

ii. $F'_i = v$.

Condition (ii) can be expressed as

$$x_i = f_i(v). \tag{A.7}$$

Suppose we treat v as an independent variable and use (A.7) with the extended definition of $f_i(v)$ to compute the corresponding x_i values. Then, (A.4a) will hold since $F_i \leq v$ when (A.7) gives $x_i = 0$. So, if (A.4c) holds also, the corresponding (x_1, x_2) solves the problem.

To find a solution to the problem from the Kuhn-Tucker conditions, we may use any real value of v. So we attempt to find a value of v such that (A.4c) holds. Let

$$f(v) = f_1(v) + f_2(v). \tag{A.8}$$

If we can find v such that f(v) = X, the corresponding (x_1, x_2) given by (A.7) is a solution to the problem. Since $f_i(v)$ are continuous monotone decreasing functions on the intervals $(0, F'_i(0)]$ with

$$f_i(F'_i(0)) = 0, \text{ and}$$

$$\lim_{v \to 0} f_i(v) = \infty,$$

f(v) is a continuous monotone decreasing function on the interval $(0, \max_i\{F'_i(0)\}]$ with

$$f(\max_i F_i'(0)) = 0, \text{ and}$$

$$\lim_{v \to 0} f(v) = \infty.$$

Therefore, there exists a unique value, v^*, satisfying $f(v) = X$. Then

$$x_i^* = f_i(v^*) \tag{A.9}$$

solves the problem. (Since the Kuhn-Tucker conditions are necessary as well as sufficient for problems in which the feasible region has an interior point, the unique Kuhn-Tucker point is the unique solution to the problem.)

In performing computations based on the above it is convenient to assign indices such that $F_1'(0) \geq F_2'(0)$. Then, the solution is:

i. If $X \leq f_1(F_2'(0))$, $x_1^* = X$, $x_2^* = 0$.

ii. If $f_1(F_2'(0)) < X$, there exists $v^* < F_2'(0)$ such that $f(v^*) = X$. Then, (A.9) gives the solution.

Application to Koopman Search Problem of Chapter 2

The Koopman search allocation problem of Chapter 2 is

Maximize $p_1(1-e^{-k_1T_1}) + p_2(1-e^{-k_2T_2}) = U$

Subject to $T_1, T_2 \geq 0$, $T_1 + T_2 \leq T$.

This is of the form of the problem analyzed above with

i. T_i replacing x_i,

ii. T replacing X, and

iii. $F_i(T_i) = p_i(1-e^{-k_iT_i})$.

So

$$F_i'(T_i) = p_i k_i e^{-k_i T_i}. \qquad \text{(A.10)}$$

which satisfy conditions (A.1a) to (A.1c). Therefore, the algorithm applies. The extended inverse marginal return functions, $f_i(v)$, are

$$f_i(v) = \begin{cases} 0 & \text{if } p_i k_i \leq v \\ \dfrac{1}{k_i} \ln \dfrac{p_i k_i}{v} & \text{if } v < p_i k_i \end{cases} \qquad \text{(A.11)}$$

Let T^s be the highest value of T for which $T_2^* = 0$. The value of T^s can be computed using (A.7) and (A.11) with $v = p_2 k_2$.

$$\begin{aligned} T^s &= f_1(p_2 k_2) + f_2(p_2 k_2) \\ &= \frac{1}{k_1} \ln \frac{p_1 k_1}{p_2 k_2}. \end{aligned} \qquad \text{(A.12)}$$

If $T \leq T^s$, the optimal allocation is $T_1^* = T$; if $T^s < T$, the optimal allocation has positive T_2^* as well as T_1^*. To compute the optimal allocation for $T^s < T$ we need to find the value of v corresponding to the optimal allocation of T. For the optimal allocation $f(v) = T$, or using (A.11)

$$\frac{1}{k_1} \ln \frac{p_1 k_1}{v} + \frac{1}{k_2} \ln \frac{p_2 k_2}{v} = T.$$

Rearranging this yields

$$(-\ln v) = \frac{T - \left[\dfrac{\ln p_1k_1}{k_1} + \dfrac{\ln p_2k_2}{k_2}\right]}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}$$

or

$$v = \exp\left\{- \frac{T - \left[\dfrac{\ln p_1k_1}{k_1} + \dfrac{\ln p_2k_2}{k_2}\right]}{\left[\dfrac{1}{k_1} + \dfrac{1}{k_2}\right]}\right\}. \tag{A.13}$$

Using this we can transform (A.7) and (A.11) into a direct expression of the allocation in terms of T (for $T^s < T$).

$$T_i^* = \frac{1}{k_i}\left\{\ln p_ik_i + \frac{T - \left[\dfrac{\ln p_1k_1}{k_1} + \dfrac{\ln p_2k_2}{k_2}\right]}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}\right\}. \tag{A.14}$$

The optimal objective function corresponding to this solution is

$$Z^* = \begin{cases} p_1(1-e^{-k_1T}) & \text{for } T \leq T^s \\ 1-\left[\dfrac{1}{k_1}+\dfrac{1}{k_2}\right] \exp\left\{- \dfrac{T - \dfrac{\ln p_1k_1}{k_1} - \dfrac{\ln p_2k_2}{k_2}}{\dfrac{1}{k_1} + \dfrac{1}{k_2}}\right\} & \text{for } T^s \leq T. \end{cases} \tag{A.15}$$

Two characteristics of this solution are worthy of note:

1. The optimal allocation follows a piecewise linear path as T increases. For $T^s < T$ the marginal allocations are inversely proportional to the rate parameters, k_i.

2. The optimal objective consists of piecewise asymptotic exponential functions.

APPENDIX B

AN ALGORITHM FOR THE GUESS PLAN CONSTRAINED SEARCH ALLOCATION PROBLEM OF CHAPTER 3

This appendix develops an efficient algorithm, suitable for machine computation, for solving problems of the following type:

$$\text{Maximize } Z = p_1 - p_1 e^{-s_1}(1-e^{-\beta_2 s_2})$$

$$+ p_2 e^{-\beta_1 s_1}(1-e^{-s_2})$$

$$\text{Subject to: } s_1, s_2 \geq 0, \; k_2 s_1 + k_1 s_2 \leq S,$$

where p_1, p_2, β_1, β_2, k_1, k_2 and S are strictly positive. [For the problem of Chapter 3 $s_i = k_i T_i$, $S = k_1 k_2 T$.]

Before proceeding with the solution to this problem, let us establish some elementary relations which will be needed later.

Let $0 < \beta < 1$ and

$$R(x) = e^{-(1 - \beta)x},$$

$$Q(x) = \frac{1 - e^{x}}{1 - e^{-\beta x}} .$$

And let the derivatives of these functions be denoted by $R'(x)$ and $Q'(x)$.

First,

$$\lim_{x \to 0} R(x) = R(0) = 1, \tag{B.1a}$$

$$\lim_{x \to \infty} R(x) = 0. \tag{B.1b}$$

And

$$R'(x) = -(1 - \beta)R(x) < 0 \text{ for all } x < \infty. \tag{B.1c}$$

Also, l'Hopital's rule implies that

$$\lim_{x \to 0} Q(x) = \frac{1}{\beta} > 1. \tag{B.2a}$$

And

$$\lim_{x \to \infty} Q(x) = 1. \tag{B.2b}$$

Consider the monotomicity of $Q(x)$. $Q(x)$ is a continuous, differentiable function over the interval $(0,\infty)$. Rolle's theorem of elementary calculus implies that either

i. $Q(x)$ is monotonic in the interval $(0,\infty)$, or

ii. there exists x_o in $(0,\infty)$ such that $Q'(x_o) = 0$.

Suppose the latter is the case.

$$Q'(x_o) = Q(x_o)\left[\frac{e^{-x_o}}{1-e^{-x_o}} - \frac{\beta e^{-\beta x_o}}{1-e^{-\beta x_o}}\right] .$$

Since $Q(x_o) > 0$, this equation can hold only if the expression in brackets is zero. Rearranging this condition, we have

$$e^{\beta x_o} - 1 = \beta(e^{x_o} - 1) .$$

Expanding this in Taylor series yields

$$(\beta-\beta^2)\frac{x_o^2}{2!} + (\beta-\beta^3)\frac{x_o^3}{3!} + \ldots = 0. \tag{B.3}$$

In this equation the coefficients of x_o^n, are non-negative with strictly positive coefficients for $n > 1$. Therefore, this equation has no positive roots. This contradicts the hypothesis that there exists x_o in $(0,\infty)$ such that $Q'(x_o) = 0$. Consequently,

$$Q'(x) < 0 \text{ for all } 0 < x < \infty. \tag{B.4}$$

Also, since the left member of (B.3) is strictly positive for $0 < x_o < \infty$, reversing the manipulations leading to (B.3) yields

$$\frac{1-e^{-x}}{1-e^{-\beta x}} > \frac{1}{\beta} e^{-(1-\beta)x} \quad \text{for all } x > 0. \tag{B.5}$$

Returning to the nonlinear programming problem at hand we have

$$Z_1 = p_1 e^{-s_1}(1-e^{-\beta_2 s_2}) - \beta_1 p_2 e^{-\beta_1 s_1}(1-e^{-s_2}) \tag{B.6}$$

and

$$Z_2 = -\beta_2 p_1 e^{-s_1-\beta_2 s_2} + p_2 e^{-\beta_1 s_1 - s_2}, \tag{B.7}$$

where Z_i denotes the partial derivative of Z with respect to s_i. From (B.6) it is clear that $Z_1 > 0$ if and only if

$s_2 \neq 0$, and (B.8a)

$$\frac{1}{\beta_1} e^{-(1-\beta_1)s_1} > \frac{p_2}{p_1} \frac{1-e^{-s_2}}{1-e^{-\beta_2 s_2}} \quad . \tag{B.8b}$$

Therefore, (B.1), (B.2) and (B.4) imply:

i. If $\frac{1}{\beta_1} \leq \frac{p_2}{p_1}$, $Z_1 < 0$ for all $0 < s_1 < \infty$, $0 < s_2 < \infty$. Consequently, there exists a solution point along the s_2-axis.

ii. If $\frac{\beta_2}{\beta_1} \geq \frac{p_2}{p_1}$, for any $s_2 > 0$ there exists $\bar{s}_1 > 0$ such that $Z_1 \overset{\geq}{\underset{<}{}} 0$ as $s_1 \overset{<}{\underset{>}{=}} \bar{s}_1$.

iii. If $\frac{\beta_2}{\beta_1} < \frac{p_2}{p_1} < \frac{1}{\beta_1}$ there exists $\bar{\bar{s}}_2 > 0$ such that $Z_1 = 0$ at $(0,\bar{\bar{s}}_2)$. $Z_1 < 0$ at (s_1, s_2) for $s_1 > 0$, $0 < s_2 \leq \bar{\bar{s}}_2$. Further for $s_2 > \bar{\bar{s}}_2$ there exists $\bar{s}_1 \geq 0$ such that

$$Z_1 \overset{>}{\underset{>}{=}} 0 \text{ as } s_1 \overset{<}{\underset{>}{=}} \bar{s}_1 .$$

Also, (B.2b), (B.4), (B.8) imply that $Z_1 < 0$ for all $s_2 > 0$ if

$$s_1 \geq \frac{\ln \frac{p_1}{\beta_1 p_2}}{1-\beta_1} \quad . \tag{B.9}$$

Finally, from (B.7) we have

$$Z_2 \overset{>}{\underset{<}{=}} 0 \text{ as}$$

$$(1-\beta_1)s_1 - (1-\beta_2)s_2 \overset{>}{\underset{<}{=}} \ln \frac{\beta_2 p_1}{p_2} \quad . \tag{B.10}$$

Suppose that (s_1^+, s_2^+), an interior point of the feasible region (the triangle $s_1 \geq 0$, $s_2 \geq 0$, $k_2 s_1 + k_1 s_2 \leq S$), is optimal. Then, $Z_1 = Z_2 = 0$ at (s_1^+, s_2^+). So, from (B.6) we have

$$\frac{p_1}{p_2} e^{-(1-\beta_1)s_1^+} = \beta_1 \frac{1-e^{-s_2^+}}{1-e^{-\beta_2 s_2^+}} , \tag{B.11a}$$

and from (B.7) we have

$$\frac{p_1}{p_2} e^{-(1-\beta_1)s_1^+} = \frac{1}{\beta_2} e^{-(1-\beta_2)s_2^+} . \tag{B.11b}$$

Eliminating s_1^+ from (B.11a) and (B.11b) yields

$$F(s_2^+) \equiv \beta_1 \beta_2 \frac{e^{s_2^+}-1}{e^{\beta_2 s_2^+}-1} = 1. \tag{B.12}$$

Note that $F(s_2^+)$ is a continuous monotonic increasing function mapping the interval $[0,\infty)$ onto the interval $[\beta_1,\infty)$. Therefore (B.12) has exactly one positive root. Corresponding to this root, s_2^+, the value of s_1^+ can be computed from (B.11a) or (B.11b). Then $Z_1 = Z_2 = 0$ at (s_1^+, s_2^+).

From (B.8) and (B.10) it is clear that the point (s_1^+, s_2^+) is a relative maximum point for Z. Then, since Z is differentiable and (s_1^+, s_2^+) is unique, this point is the unique global maximum point for Z. That is, (s_1^+, s_2^+) is the unique solution to the problem if it is feasible. And (s_1^+, s_2^+) is feasible if $s_1^+ \geq 0$, $k_2 s_1^+ + k_1 s_2^+ \leq S$.

If (s_1^+, s_2^+) is not feasible, the solution to the problem must be on the boundary of the feasible region. Then, the solution point may be characterized as a point where the gradient of the objective function (the vector (Z_1,Z_2)) is either the null vector or an outward normal to a support line of the feasible region.

Let

$\theta(s_1,s_2)$ = the angle between the gradient vector (Z_1, Z_2) and the positive s_1 axis with positive angles representing counter-clockwise rotation.

Then

$$\theta(s_1, s_2) = \begin{cases} \tan^{-1} \dfrac{Z_2(s_1,s_2)}{Z_1(s_1,s_2)} & \text{if } Z_1 > 0, \\ -\tan^{-1} \dfrac{Z_2(s_1,s_2)}{Z_1(s_1,s_2)} & \text{if } Z_1 < 0. \end{cases} \tag{B.13}$$

Consider $\theta(s_1, s_2)$ for (s_1, s_2) on the boundary of the feasible region. First, by (B.8) and (B.10)

$$\theta(0,0) = \begin{cases} +90° & \text{if } p_2 > \beta_2 p_1, \\ \text{undefined } ((Z_1,Z_2) = (0,0)) & \text{if } p_2 = \beta_2 p_1, \\ -90° & \text{if } p_2 < \beta_2 p_1. \end{cases} \tag{B.14a}$$

Similarly, along the positive s_1 axis

$$\theta(s_1,0) = \begin{cases} +90° & \text{if } s_1 > c/(1-\beta_1), \\ \text{undefined} & \text{if } s_1 = c/(1-\beta_1), \\ -90° & \text{if } s_1 < c/(1-\beta_1), \end{cases} \tag{B.14b}$$

where

$$c = \ln \frac{\beta_2 p_1}{p_2} \quad .$$

Along the s_1 axis the outward normal support line of the feasible region has direction $\theta_n = -90°$. Therefore, a point on the s_1 axis can solve the problem only if $p_2 < \beta_2 p_1$. Further, since $Z_1 = 0$ for all points on the s_1 axis, all feasible points on the s_1 axis yield the same objective function value, p_1. But this value of the objective function can be attained in the decision maker's real problem without searching at all or by ignoring the search information in making the response guess decision. Thus, if a solution to the problem lies on the s_1 axis, all feasible search allocations are optimal for the decision maker's real problem.

Now consider the behavior of θ along the searching time constraint line $k_2 s_1 + k_1 s_2 = S$ in the positive quadrant. We will need the second partial derivatives of Z with respect to s_1 and s_2.

$$Z_{11} = -p_1 e^{-s_1}(1-e^{-\beta_2 s_2}) + \beta_1^2 p_2 e^{-\beta_1 s_1}(1-e^{-s_2}), \qquad \text{(B.15a)}$$

$$Z_{12} = \beta_2 p_1 e^{-s_1-\beta_2 s_2} - \beta_1 p_2 e^{-\beta_1 s_1 - s_2}, \qquad \text{(B.15b)}$$

$$Z_{22} = \beta_2^2 p_1 e^{-s_1-\beta_2 s_2} - p_2 e^{-\beta_1 s_1 - s_2} . \qquad \text{(B.15c)}$$

Since $\beta_1, \beta_2 < 1$, $Z_{11} < -Z_1$ and $Z_{22} < -Z_2$. Therefore, for the region in which $Z_1 > 0$ and $Z_2 > 0$ both Z_{11} and Z_{22} are

negative. Suppose Z_{12} is negative. Then, by (B.5)

$$\frac{p_1}{p_2} e^{-(1-\beta_1)s_1} < \beta_1 \frac{1-e^{-s_2}}{1-e^{-\beta_2 s_2}} ,$$

which implies that $Z_1 < 0$ (by B.6). So Z_{12} is positive while Z_{11} and Z_{22} are negative for the region in which Z_1 and Z_2 are nonnegative. Along a line $k_2 s_1 + k_1 s_2 = S$.

$$\frac{dZ_1}{ds_1} = \frac{\partial Z_1}{\partial s_1} + \frac{\partial Z_1}{\partial s_2}\frac{ds_2}{ds_1}$$

$$= Z_{11} - \frac{k_2}{k_1} Z_{12}, \text{ and} \tag{B.16a}$$

$$\frac{dZ_2}{ds_1} = \frac{\partial Z_2}{\partial s_1} + \frac{\partial Z_2}{\partial s_2}\frac{ds_2}{ds_1}$$

$$= Z_{12} - \frac{k_2}{k_1} Z_{22}. \tag{B.16b}$$

Therefore, in the region where Z_1 and Z_2 are nonnegative, Z_1 is decreasing while Z_2 is increasing as s_1 increases along $k_2 s_1 + k_1 s_2 = S$. Since the support line to the feasible region along such a line is the line itself, the outward normal along $k_2 s_1 + k_1 s_2 = S$ has direction $\theta_o = \tan^{-1}\frac{k_2}{k_1}$. Because $0 < \theta_o < 90°$, both Z_1 and Z_2 must be positive at any solution along the positive quadrant segment of $k_2 s_1 + k_1 s_2 = S$. Therefore, there exists at most one solution point with $s_1, s_2 > 0$ on $k_2 s_1 + k_1 s_2 = S$;

this point (if it exists) is characterized by

$$k_2 Z_2 = k_1 Z_1 . \tag{B.17}$$

Finally, consider the behavior of θ along the s_2 axis. The outward normal to the support line of the feasible region at $(0, s_2)$ for $0 < s_2 < S/k_1$ has direction $\theta_n = 180°$. Therefore, any solution point in this interval must satisfy:

i. $Z_2 = 0$, and (B.18a)

ii. $Z_1 \leq 0$. (B.18b)

Condition (i) is

$$s_2 = \frac{-c}{1-\beta_2} ; \tag{B.19a}$$

condition (ii) is

$$\frac{p_1}{\beta_1 p_2} \leq \frac{1-e^{-s_2}}{1-e^{-\beta_2 s_2}} \tag{B.19b}$$

(B.19a) establishes the uniqueness of possible solutions in the $0 < s_2 < S/k_1$ segment of the s_2 axis.

In addition to points on the three sides of the feasible region, the three corners of the feasible region must be considered as possible boundary solutions to the problem. The origin satisfies the marginal (normal objective function gradient) necessary condition for solving the problem if $\beta_2 p_1 > p_2$. But, the origin only solves the

decision maker's real problem if no possible search can achieve a higher objective function value than that achieved by not searching at all. (This case does occur for small amounts of available searching time if $\beta_2 p_1 > p_2$.) The corner point $(S/k_2, 0)$ yields the same value of objective function, p_1, as the origin. Thus, this point can only be optimal when the origin is also optimal. Finally, the support lines of the feasible region at $(0, S/k_1)$ have normal directions, θ_n, for each angle in the interval from θ_o to 180°. Therefore, if $\theta(0, s/k_1)$ is in the interval $[\theta_o, 180°]$, the point $(0, s/k_1)$ satisfies the marginal necessary condition for optimality.

We shall now establish that, except for the case in which all feasible points solve the decision maker's real problem, the solution to the problem is unique.

Suppose that there exists a potential solution point $(0, s_2^c)$ with s_2^c in the segment $(0, S/k_1)$ satisfying the gradient necessary condition for optimality. Then (B.18a) and (B.10) imply that $Z_2 < 0$ at $(0, S/k_1)$ so $(0, S/k_1)$ cannot be a solution. Now suppose there exists an optimal solution, (s_1^o, s_2^o), in the positive quadrant segment of $k_2 s_1 + k_1 s_2 = S$. Then, since $Z_2 > 0$ at (s_1^o, s_2^o), the point (s_1^o, s_2^o) lies below the $Z_2 = 0$ line. And since $Z_1 > 0$, this point lies above the $Z_1 = 0$ curve. But the point $(0, s_2^c)$ which is on the $Z_2 = 0$ line must lie below the $Z_1 = 0$ curve, and the $Z_1 = 0$ curve lies above the

$Z_2 = 0$ line for large values of s_1. Therefore, the $Z_1 = 0$ curve must cross the $Z_2 = 0$ line at least twice at points where $s_2 > 0$. This contradicts the uniqueness of the solution, s_2^+, of (B.12) in the positive half plane. Therefore, the point (s_1^o, s_2^o) cannot exist. To establish the uniqueness of the solution point $(0, s_2^c)$ then, we need only observe that by (B.10) $Z_2 > 0$ for all points $(0, s_2)$ with s_2 in the interval $(0, s_2^c)$ so that at $(0, s_2^c)$ $Z > p_1$.

Now suppose that there exists a potential solution point (s_1^c, s_2^c) in the positive quadrant portion segment of $k_2s_1 + k_1s_2 = S$. Then, the reasoning of the previous paragraph implies that no potential solution satisfying the first order marginal necessary condition can exist along the s_2 axis with $k_2s_1 + k_1s_2 < S$. To establish the uniqueness of (s_1^c, s_2^c) as the solution point, then, we need only observe that $Z_2 > 0$ at (s_1^c, s_2^c) so that, by (B.10), $Z_2 > 0$ for all (s_1^c, s_2) with s_2 in $(0, s_2^c)$. Therefore, at (s_1^c, s_2^c) $Z > p_1$, and no solution point exists along the s_1 axis.

To complete the proof that solutions which are not on the s_1 axis are unique we need to show that corner solutions $(0, S/k_1)$ are incompatible with (other) solutions along the s_2 axis and the line $k_2s_1 + k_1s_2 = S$. First, if $(0, S/k_1)$ satisfies the gradient necessary condition for boundary solutions, $Z_2 \geq 0$ at $(0, S/k_1)$. Then, by (B.10)

$Z_2 > 0$ for all other feasible points on the s_2 axis. Therefore, no other potential solution point can exist on the s_2 axis. Suppose that $Z_1 \leq 0$ at $(0, S/k_1)$. Then, by (B.8) $Z_1 \leq 0$ for all $(s_1, S/k_1)$ with $s_1 > 0$. And consequently, (B.4) and (B.8) together imply that $Z_1 \leq 0$ for all (s_1, s_2) with $s_1 > 0$, $0 \leq s_2 \leq S/k_1$. That is, $Z_1 \leq 0$ for all feasible points. Thus, no (other) solution point along $k_2s_1 + k_1s_2 = S$ can exist. Now suppose that $Z_1 > 0$ at $(0, S/k_1)$. Then, (B.16a) and(B.16b) imply that $\theta > \tan^{-1}k_2/k_1$ for all points along $k_2s_1 + k_1s_2 = S$ having s_1, $s_2 > 0$, $Z_1 > 0$ and $Z_2 > 0$. Hence, no (other) potential solution point exists along $k_2s_1 + k_1s_2 = S$. This establishes the uniqueness of solutions to the problem at hand except when all feasible points solve the decision maker's real problem.

<u>The Algorithm</u>

The above development leads to the following algorithm for computing approximate numerical solutions to the problem:

1. Determine if the line $Z_2 = 0$ and the curve $Z_1 = 0$ intersect in the first quadrant. They fail to intersect if and only if

$$\beta_2 < \frac{P_2}{P_1} , \qquad \text{and}$$

$$\frac{1-\left[\frac{\beta_2 P_1}{P_2}\right]^{\frac{\beta_2}{1-\beta_2}}}{1-\left[\frac{\beta_2 P_1}{P_2}\right]^{\frac{1}{1-\beta_2}}} \le \frac{\beta_1 P_2}{P_1} \quad . \tag{B.20}$$

2a. If $Z_2 = 0$ and $Z_1 = 0$ fail to intersect in the first quadrant, the solution is

$$\underline{s}^* = \left(0, \min\left\{S/k_1, \frac{\ln \frac{P_2}{\beta_2 P_1}}{1-\beta_2}\right\}\right) \quad . \tag{B.21}$$

2b. If $Z_2 = 0$ and $Z_1 = 0$ intersect in the first quadrant, compute the coordinates of the point of intersection, $\underline{s}^+$, as follows:

(1) Compute the intercept, s_1'', of the line asymptotic to $Z_1 = 0$ from

$$s_1'' = \frac{\ln \frac{P_1}{\beta_1 P_2}}{1-\beta_1} \quad .$$

Then compute the value, s_2'', at which $Z_2 = 0$ crosses this intercept.

$$s_2'' = \frac{-\ln(\beta_1\beta_2)}{1-\beta_2} \quad . \tag{B.22a}$$

Also, compute

$$s_2' = \frac{1}{1-\beta_2} \max\left\{-\ln\beta_1, \ln \frac{P_2}{\beta_2 P_1}\right\} \quad . \tag{B.22b}$$

(2) Using s_2' and s_2'' as bounds for s_2^+ compute[1] the value of s_2^+ from (B.12).

(3) Compute the corresponding value s_1^+ from $Z_2 = 0$ using (B.11b).

3a. If $k_2 s_1^+ + k_1 s_2^+ \le S$, the optimal solution is $\underline{s}^* = (s_1^+, s_2^+)$.

3b. If $k_2 s_1^+ + k_1 s_2^+ > S$, the solution lies somewhere on the boundary of the feasible triangle.

4. (1) Compute S_1^o, the value of S which must be attained before any search effort can be allocated to box 2.

$$s_1^o = k_2 \max \left\{ \frac{\ln \frac{\beta_2 p_1}{p_2}}{1-\beta_1} , \quad 0 \right\} . \tag{B.23a}$$

If $0 < S \le S_1^o$, the solution to the problem is $(s_1^*, 0)$ for any $0 \le s_1^* \le S/k_2$. The optimal value of the objective function, Z^*, is p_1. If $S_1^o < S$, the solution to the problem does not lie on the s_1 axis.

(2) Compute S_2^o, the value S must exceed if a positive amount of search effort is to be allocated to box 1.

$$\text{If } \beta_2 \ge \frac{p_2}{p_1} , \; S_2^o = 0. \tag{B.23b}$$

[1]Several numerical techniques exist for performing this computation on a modern digital computer.

If $\beta_2 < \frac{P_2}{P_1}$, S_2^o corresponds to the point on the s_2 axis at which $k_1Z_1 = k_2Z_2$. This point can be found by finding the root of (B.24) in the interval

$$\left(0, \frac{\ln \frac{P_2}{\beta_2 P_1}}{1-\beta_2}\right).$$

$$\left(1-\beta_1\frac{P_2}{P_1}\right) + \left(\beta_2\frac{k_2}{k_1} - 1\right)e^{-\beta_2 x} + \frac{P_2}{P_1}\left(\beta_1-\frac{k_2}{k_1}\right)e^{-x} = 0 \qquad \text{(B.24)}$$

Then $S_2^o = k_1x$.

If $0 < s \leq S_2^o$, $(0, S/k_1)$ solves the problem.

If $S_2^o < S$, the solution lies on the searching time constraint line, $k_2s_1 + k_1s_2 = S$.

5. Let $s_2' = 0$, $s_2'' = S/k_1$ be bounds on the s_2^*, the s_2 coordinate of the optimal solution point. s_2^* is the root of equation B.25, which expresses $k_2s_1 + k_1s_2 = S$, $k_1Z_1 = k_2Z_2$.

$$k_2\left\{-\beta_2P_1e^{-\left(\frac{S}{k_1}-s_2\right)/k_2}e^{-\beta_2s_2} + P_2e^{-\beta_1\left(\frac{S}{k_1}-s_2\right)/k_2}e^{-s_2}\right\} \qquad \text{(B.25)}$$

$$= k_1\left\{P_1e^{-\left(\frac{S}{k_1}-s_2\right)/k_2}\left(1-e^{-\beta_2s_2}\right) - \beta_1P_2e^{-\beta_1\left(\frac{S}{k_1}-s_2\right)/k_2}\left(1-e^{-s_2}\right)\right\}$$

The solution to the problem is (s_1^*, s_2^*), where s_2^* is the root of (B.25) and $s_1^* = (S - k_1 s_2^*)/k_2$.

APPENDIX C

BEHAVIOR OF SR-LSI MODEL FOR SMALL T

For small T approximate solutions to the SR-LSI model can be obtained by replacing the objective function by the first few terms of a Taylor series expansion of the objective and maximizing this approximate objective function over the feasible set. The Taylor series expansion of the objective function for guess plan I (the objective function of Appendix B) is approximately

$$Y = p_1 + (p_2-\beta_2p_1)k_2T_2 + (\beta_2p_1-\beta_1p_2)k_1k_2T_1T_2 - (p_2-\beta_2^2p_1)k_2^2\frac{T_2^2}{2} \quad . \tag{C.1}$$

This approximation differs from the exact objective function only in third and higher order terms. We wish to maximize Y subject to $T_1, T_2 \geq 0$, $T_1 + T_2 \leq T$.

If T is small and $p_2 \neq \beta_2p_1$, the linear term of (C.1) dominates the quadratic and higher order terms. Therefore, the solution for sufficiently small T assuming $p_2 \neq \beta_2p_1$ is:

i. $T_2^* = 0$, $0 \leq T_1^* \leq T$ if $p_2 < \beta_2p_1$.
(In Appendix B we found that the objective function is constant along the T axis.)

ii. $T_1^* = 0$, $T_2^* = T$ if $p_2 > \beta_2 p_1$.

If T is small and $p_2 = \beta_2 p_1$, the linear term of (C.1) is zero. So the solution is determined by the quadratic terms. (C.1) reduces to

$$Y = p_1 + BT_1T_2 - CT_2^2 , \tag{C.2}$$

where

$$B = (\beta_2 p_1 - \beta_1 p_2)k_1k_2$$

$$C = \frac{1}{2}(p_2 - \beta_2^2 p_1)k_2^2 .$$

Since $p_2 = \beta_2\ _1$ and $\beta_2 < 1$, B and C are both positive. To maximize Y given by (C.2) over the feasible set $T_1 = T-T_2$ since $\frac{\partial Y}{\partial T_1} = B > 0$. Let us substitute $T_1 = T-T_2$ in (C.2) and then maximize Y as a function of one variable, T_2.

$$Y = p_1 + BTT_2 - (B+C)T_2^2 \tag{C.3}$$

(C.3) is the equation of a parabola having positive slope at $T_2 = 0$. Clearly Y is maximized by

$$T_2^* = \min\{T,\ T_2^o \text{ such that } \frac{dY}{dT_2} = 0\} . \tag{C.4}$$

T_2^o is easily found by differentation of (C.3) to be

$$T_2^o = \frac{BT}{2(B+C)} . \tag{C.5}$$

Since B and C are positive, $T_2^o < T$. Therefore, (C.4) becomes

$$T_2^* = T_2^o = \frac{BT}{2(B+C)} . \tag{C.6a}$$

Therefore,

$$T_1^* = T - T_2^* = \frac{(B+2C)T}{2(B+C)} . \qquad \text{(C.6b)}$$

Having determined the solution of the guess plan constrained optimal search allocations for small T, let us consider which of these two conditionally optimal allocations solves the decision maker's search problem. The solution given above is conditionally optimal given that the guess plan is: "Guess that the target is in box 1 unless a contact occurs in box 2 only." Let us call this solution conditional solution 1 and the corresponding optimal approximate objective function Y^1. The other candidate-optimal guess plan and its objective function are of those same forms with the roles of boxes (subscripts in formulas) interchanged. Therefore, conditional solution 2 is given by the above development with subscripts interchanged.

Let i be the number of one of the conditional solutions while j is the number of the other, i.e., j = 1-3.

<u>Case 1</u> There exists i such that $\beta_i p_j > p_i$. The conditional solution i is

$$T_i^* = 0, \ T_j^* = T ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^i = p_i + (p_j - \beta_j p_i) k_j T.$$

Similarly, the conditional solution j is

$$T_i^* = 0, \ 0 \leq T_j^* \leq T \ ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^j = p_j.$$

Since $\beta_i p_j > p_i$ implies that $p_j > p_i$, the solution to the decision maker's problem for small T is conditional solution j. But this solution yields exactly the same objective function value as would be achieved without a search. This value can be attained for any feasible search allocation by the degenerate guess plan: Guess that the target is in box j for any possible search outcome. Thus all feasible points are solutions to the decision maker's allocation problem.

Case 2 There exists i such that $\beta_i p_j = p_i$. The conditional solution i is

$$T_i^* = 0, \ T_j^* = T \ ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^i = p_i + (p_j - \beta_j p_i)T.$$

The conditional solution j is

$$T_i^* + \frac{(\beta_i p_j - \beta_j p_i)k_j}{2(\beta_i p_j - \beta_j p_i)k_j + (p_i - \beta_i^2 p_j)k_i} T ,$$

$$T_j^* = \frac{(\beta_i p_j - \beta_j p_i)k_j + (p_i - \beta_i^2 p_j)k_i}{2(\beta_i p_j - \beta_j p_i)k_j + (p_i - \beta_i^2 p_j)k_i} T ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^j = p_j + (\beta_i p_j - \beta_j p_i)k_j k_i T_i^* T_j^* - \frac{1}{2}(p_i - \beta_i^2 p_j)k_i^2 T^*{}_i{}^2 .$$

Since $\beta_i p_j = p_i$ and $\beta_i < 1$, $Y^j > Y^i$. That is, the conditional solution j is optimal.

<u>Case 3</u> $p_1 > \beta_1 p_2$, $p_2 > \beta_2 p_1$, $p_i < p_j$. The conditional solution i is

$$T_i^* = 0,\ T_j^* = T ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^i = p_i + (p_j - \beta_j p_i)k_j T.$$

Similarly, the conditional solution j is

$$T_i^* = T,\ T_j^* = 0 ,$$

and the corresponding conditional optimal objective function is

$$Y^j = p_j + (p_i - \beta_i p_j)k_i T.$$

Since $p_i < p_j$, $Y^i < Y^j$ for small T. That is, conditional solution j is optimal.

Case 4 $\quad p_1 = p_2 = .5$ The conditional solution i is

$$T_i^* = 0,\ T_j^* = T\ ,$$

and the corresponding conditional optimal objective function is approximately

$$Y^i = .5[1 + (1 - \beta_j)k_j T - \tfrac{1}{2}(1 - \beta_j^2)k_j^2 T^2].$$

Similarly, the conditional solution j is

$$T_i^* = T,\ T_j^* = 0\ ,$$

and the corresponding conditional optimal objective function is

$$Y^j = .5[1 + (1 - \beta_i)k_i T - \tfrac{1}{2}(1 - \beta_i^2)k_i^2 T^2].$$

i. If $(1-\beta_j)k_j > (1-\beta_i)k_i$, $Y^i > Y^j$ and conditional solution i is optimal.

ii. If $(1-\beta_1)k_1 = (1-\beta_2)k_2$ and $(1-\beta_i^2)k_i^2 < (1-\beta_j^2)k_j^2$, $Y^j > Y^i$ and conditional solution j is optimal.

iii. If $(1-\beta_1)k_1 = (1-\beta_2)k_2$ and $(1-\beta_1^2)k_1^2 = (1-\beta_2^2)k_2^2$, then $\beta_1 = \beta_2$ and $k_1 = k_2$ so that both conditional solutions yield the same conditional optimal objective functions. Hence both are optimal.

To see that

(i) $p_1 = p_2$,

(ii) $(1-\beta_1)k_1 = (1-\beta_2)k_2$, and

(iii) $(1-\beta_1^2)k_1^2 = (1-\beta_2^2)k_2^2$

imply that $\beta_1 = \beta_2$ and $k_1 = k_2$, eliminate k_1 and k_2 from (ii) and (iii). This yields

$$\frac{(1-\beta_1)^2}{1-\beta_1^2} = \frac{(1-\beta_2)^2}{1-\beta_2^2} .$$

Numerator and denominator of each side have common factors $(1-\beta_i)$. So we have

$$\frac{1-\beta_1}{1+\beta_1} = \frac{1-\beta_2}{1+\beta_2} .$$

But the function $\frac{1-x}{1+x}$ is strictly monotonic in the interval $[0,1]$. Therefore, $\beta_1 = \beta_2$. Then, substituting in (ii) yields $k_1 = k_2$.

APPENDIX D

ALGORITHMS FOR SOLVING THE SR-CSI AND SR-ACSI SEARCH ALLOCATION PROBLEMS

In this appendix we develop algorithms for both the SR-CSI and SR-ACSI allocation problems.

THE SR-CSI PROBLEM

Consider first the SR-CSI problem given by

$$\text{SR-CSI}\left\{\begin{array}{l} \underset{\underline{T}}{\text{Maximize}}\ Z = \displaystyle\iint_{\underline{0}\le\underline{t}\le\underline{T}} \max_{i=1,2}\left\{p_i(\underline{t},\underline{T})\right\} d\bar{F}(\underline{t}) \\ \\ \text{Subject to } T_1,\ T_2 \ge 0,\ T_1 + T_2 \le T, \end{array}\right. \tag{D.1}$$

where

$$p_1(\underline{t},\underline{T}) = \frac{u(\underline{t},\underline{T})}{u(\underline{t},\underline{T})+1},$$

$$p_2(\underline{t},\underline{T}) = 1 - p_1(\underline{t},\underline{T}),$$

$u(\underline{t},\underline{T})$ is given by (24) of Chapter 3, and

$\bar{F}(\underline{t})$ is given by (25) of Chapter 3.

Consider the response guess decision and the resulting objective function, $Z(\infty)$, for the related search time unconstrained problem with $T_1 = \infty$ and $T_2 = \infty$. To evaluate the integral in (D.1) for $Z(\infty)$ one must evaluate the $\max_{i=1,2}\left\{p_i(\underline{t},\underline{\infty})\right\}$ function over the $\underline{t}$ plane. From (24)

of Chapter 3 and the definition of $u(\underline{t},\underline{T})$ clearly

$$p_1(\underline{t},\underline{\infty}) \gtreqless p_2(\underline{t},\underline{\infty}) \text{ as}$$

$$(1-\beta_1)k_1t_1 - (1-\beta_2)k_2t_2 \lesseqgtr \ln \frac{\beta_2 p_1}{\beta_1 p_2} . \qquad (D.2a)$$

That is,

$$G(\underline{t},\underline{\infty}) = \begin{cases} \{1\} & \text{if } s(\underline{t}) < d, \\ \{1,2\} & \text{if } s(\underline{t}) = d, \\ \{2\} & \text{if } s(\underline{t}) > d, \end{cases}$$

where

$$s(t) = (1-\beta_1)k_1t_1 - (1-\beta_2)k_2t_2,$$

$$d = \ln\left(\frac{\beta_2 p_1}{\beta_1 p_2}\right) .$$

Therefore,

$$Z(\infty) = \iint_{s(\underline{t})<d} p_1(\underline{t},\underline{\infty})d\overline{F}(\underline{t}) + \iint_{s(\underline{t})>d} p_2(\underline{t},\underline{\infty})d\overline{F}(\underline{t}) \qquad (D.3)$$

By (24) of Chapter 3

$$p_1(\underline{t},\underline{\infty}) = \frac{\beta_2 p_1 e^{-(1-\beta_1)k_1t_1}}{\beta_2 p_1 e^{-(1-\beta_1)k_1t_1} + \beta_1 p_2 e^{-(1-\beta_2)k_2t_2}} , \qquad (D.4a)$$

and

$$p_2(\underline{t},\underline{\infty}) = \frac{\beta_1 p_2 e^{-(1-\beta_2)k_2t_2}}{\beta_2 p_1 e^{-(1-\beta_1)k_1t_1} + \beta_1 p_2 e^{-(1-\beta_2)k_2t_2}} . \qquad (D.4b)$$

By (25) of Chapter 3

$$d\bar{F}(\underline{t}) = \left(\beta_2 p_1 e^{-k_1 t_1 - \beta_2 k_2 t_2} + \beta_1 p_2 e^{-\beta_1 k_1 t_1 - k_2 t_2}\right) dt_1 dt_2 \qquad \text{(D.4c)}$$

Using (D.2) and (D.4), (D.3) reduces to

$$Z(\infty) = \begin{cases} p_2 + \dfrac{(1-\beta_2)^2}{1-\beta_1\beta_2} p_1 \left[\dfrac{\beta_2 p_1}{\beta_1 p_2}\right]^{\frac{\beta_2}{1-\beta_2}} & \text{if } \beta_2 p_1 \le \beta_1 p_2 \quad \text{(D.5a)} \\[2ex] p_1 + \dfrac{(1-\beta_1)^2}{1-\beta_1\beta_2} p_2 \left[\dfrac{\beta_1 p_2}{\beta_2 p_1}\right]^{\frac{\beta_1}{1-\beta_1}} & \text{if } \beta_1 p_2 \le \beta_2 p_1 . \quad \text{(D.5b)} \end{cases}$$

If $\beta_2 p_1 = \beta_1 p_2$, both (D.5a) and (D.5b) reduce to

$$Z(\infty) = \frac{p_2 - 2\beta_2 p_1 p_2}{p_2 - \beta_2^2 p_1} \quad \text{if } \beta_1 p_2 = \beta_2 p_1 . \qquad \text{(D.5c)}$$

Now let us analyze the SR-CSI allocation problem by comparing the objective function with the related unconstrained search problem. Consider each possible unconstrained search outcome, $\underline{t}$. If $\underline{t} \le \underline{T}$, the constrained search will lead to the same response guess as the unconstrained search. But of either $t_1 > T_1$ or $t_2 > T_2$, the constrained and unconstrained searches may lead to dif-

ferent guesses. Let

$$H(\underline{T}) = Z(\infty) - Z.$$

Then, to solve the SR-CSI problem we need to minimize $H(\underline{T})$ subject to the nonnegativity and total capacity constraints on $\underline{T}$. Let

$$\tau_i = \min\{t_i, T_i\}.$$

Then,

$$H(\underline{T}) = \iint_{\underline{0} \leq \underline{t}} \left[\max_{i=1,2}\{p_i(\underline{t},\underline{\infty})\} - p_g(\underline{t},\underline{\infty})\right] d\bar{F}(\underline{t}), \qquad (D.6)$$

where g is in $G(\underline{\tau},\underline{T})$.

Let

$$W(\underline{T}) = \{\underline{t} \mid p_g(\underline{t},\underline{\infty}) < \max_{i=1,2} \{p_i(\underline{t},\underline{\infty})\}\},$$

for g in $G(\underline{\tau},\underline{T})$. Then, (D.6) can be expressed as

$$H(\underline{T}) = \iint_{W(\underline{T})} \left[p_r(\underline{t},\underline{\infty}) - p_g(\underline{t},\underline{\infty})\right] d\bar{F}(\underline{t}), \qquad (D.7)$$

where g is in $G(\underline{\tau},\underline{T})$ and $r = 3-g$. But, $\left[p_r(\underline{t},\underline{\infty}) - p_g(\underline{t},\underline{\infty})\right]$ is independent of $\underline{T}$ given that $\underline{t}$ is in $W(\underline{T})$. Therefore, (D.7) can be expressed as

$$H(\underline{T}) = \iint_{W(\underline{T})} h(\underline{t}) d\bar{F}(t), \qquad (D.8a)$$

where

$$h(\underline{t}) = \begin{cases} p_1(\underline{t},\underline{\infty}) - p_2(\underline{t},\underline{\infty}) & \text{if } s(\underline{t}) < d \\ p_2(\underline{t},\underline{\infty}) - p_1(\underline{t},\underline{\infty}) & \text{if } s(\underline{t}) > d. \end{cases} \tag{D.8b}$$

Using (D.4), (D.8) becomes

$$H(\underline{T}) = \iint\limits_{\substack{s(\underline{t})<d \\ \text{in } W(\underline{T})}} y(t_1,t_2)dt_1dt_2 - \iint\limits_{\substack{s(\underline{t})>d \\ \text{in } W(\underline{T})}} y(t_1,t_2)dt_1dt_2 \tag{D.9}$$

where

$$y(t_1,t_2) = \beta_2 p_1 e^{-(1-\beta_1)k_1t_1} - \beta_1 p_2 e^{-(1-\beta_2)k_2t_2} .$$

Note that by (D.2b) the integrand in (D.9) is strictly positive for all $\underline{t}$ in $W(\underline{T})$. Therefore, if $W(\underline{T}+)$ is a proper subset of $W(\underline{T}^\circ)$, $H(\underline{T}+) < H(\underline{T}^\circ)$ and $\underline{T}^\circ$ cannot solve the problem.

To evaluate $H(\underline{T})$ using (D.9) one must define the wrong guess region of the $\underline{t}$ plane, $W(\underline{T})$. From (24a) of Chapter 3 we see that for $\underline{\tau} = \underline{T}$,

$$G(\underline{\tau},\underline{T}) = \begin{cases} \{1\} & \text{if } s(\underline{\tau}) < a \\ \{1,2\} & \text{if } s(\underline{\tau}) = a \\ \{2\} & \text{if } s(\underline{\tau}) > a, \end{cases} \tag{D.10a}$$

where

$$a = \ln \frac{p_1}{p_2} \ .$$

From (24b) of Chapter 3 for $\tau_1 < T_1$, $\tau_2 = T_2$

$$G(\underline{\tau},\underline{T}) = \begin{cases} \{1\} & \text{if } s(\underline{\tau}) < b \\ \{1,2\} & \text{if } s(\underline{\tau}) = b \\ \{2\} & \text{if } s(\underline{\tau}) > b, \end{cases} \tag{D.10b}$$

where

$$b = \ln \frac{p_1}{\beta_1 p_2} \ .$$

From (24c) of Chapter 3 for $\tau_1 = T_1$, $\tau_2 < T_2$

$$G(\underline{\tau},\underline{T}) = \begin{cases} \{1\} & \text{if } s(\underline{\tau}) < c \\ \{1,2\} & \text{if } s(\underline{\tau}) = c \\ \{2\} & \text{if } s(\underline{\tau}) > c, \end{cases} \tag{D.10c}$$

where

$$c = \ln \frac{\beta_2 p_1}{p_2} \ .$$

And, if $\underline{\tau} < \underline{T}$, $G(\underline{\tau},\underline{T})$ is given by (D.2b) with $\underline{\tau}$ replacing $\underline{t}$. We will refer to the lines $s(\underline{t}) = a$, $s(\underline{t}) = b$, $s(\underline{t}) = c$ and $s(\underline{t}) = d$ as lines a, b, c, and d respectively.

An optimal guess plan corresponding to the unconstrained search is: "Guess that the target is in box 1 if $\underline{t}$ lies above line d; otherwise guess that the target is in box 2." Therefore, if $\underline{\tau} = \underline{t} < \underline{T}$, the constrained search

yields the same guess as the unconstrained search. Hence, no point $\underline{t} < \underline{T}$ is in $W(\underline{T})$. If $\underline{\tau} = \underline{T}$ ($\underline{t} \geq \underline{T}$), the constrained and unconstrained searches yield the same guesses for those points, $\underline{t}$, which lie above or below line d as $\underline{T}$ is above or below line a. Similarly, if $\tau_1 < T_1$ and $\tau_2 = T_2$ ($t_1 < T_1$ and $t_2 \geq T_2$), the constrained and unconstrained searches yield the same guesses for those points, $\underline{t}$, which lie above or below line d as the point (t_1, T_2) lies above or below line b. If $\tau_1 = T_1$ and $\tau_2 < T_2$ ($t_1 \geq T_1$ and $t_2 < T_2$), the constrained and unconstrained searches yield the same guesses for those points, $\underline{t}$, which lie above or below line d as the point (T_1, t_2) lies above or below line c. These conditions completely define the region $W(\underline{T})$. Figure D.1 shows the 10 possible $W(\underline{T})$ region shapes assuming that $\beta_1 < \beta_2$. The corresponding cases for $\beta_2 < \beta_1$ may be obtained by interchanging the roles of t_1 and t_2. If $\beta_1 = \beta_2$ lines a and d coincide so that cases 3 and 3' do not exist.

Since the integrand of (D.9) is strictly positive for all $\underline{t}$ in $W(\underline{T})$ and $W(\underline{T})$ has measure greater than zero for any finite $T_1 + T_2$, $H(\underline{T})$ is strictly positive for $T_1 + T_2$ finite. But, the integrand decreases exponentially to zero as T_1 and T_2 grow without bound. Therefore, $H(\underline{T})$ approaches zero if both T_1 and T_2 grow without bound. Hence, the amount of available searching time used by any

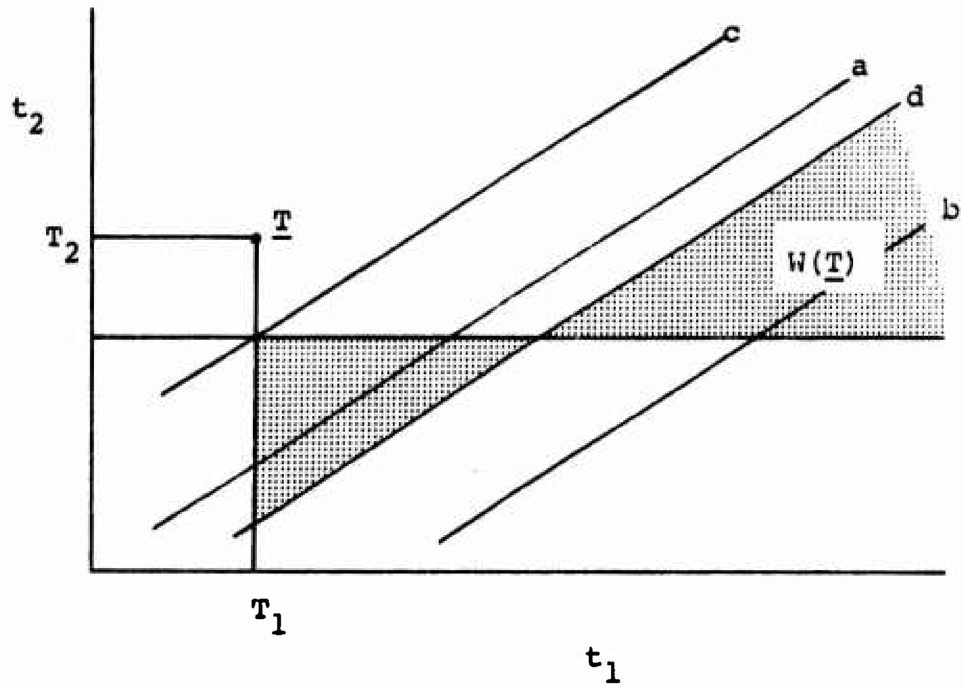

Figure D.1a SR-CSI Model $W(\underline{T})$ for $\beta_1 < \beta_2$ Case 1

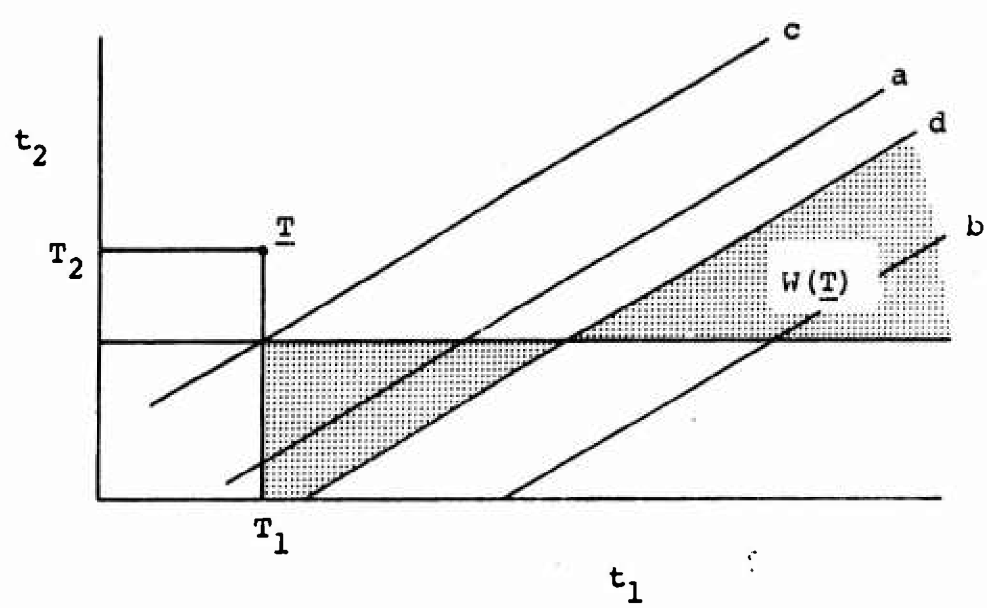

Figure D.1b SR-CSI Model $W(\underline{T})$ for $\beta_1 < \beta_2$ Case 1'

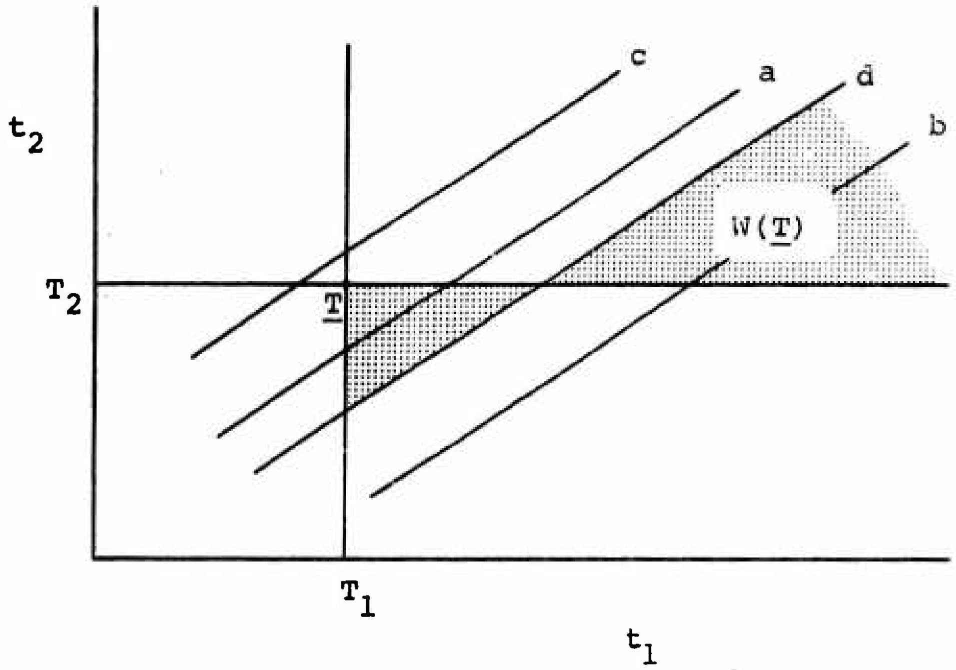

Figure D.1c SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 2

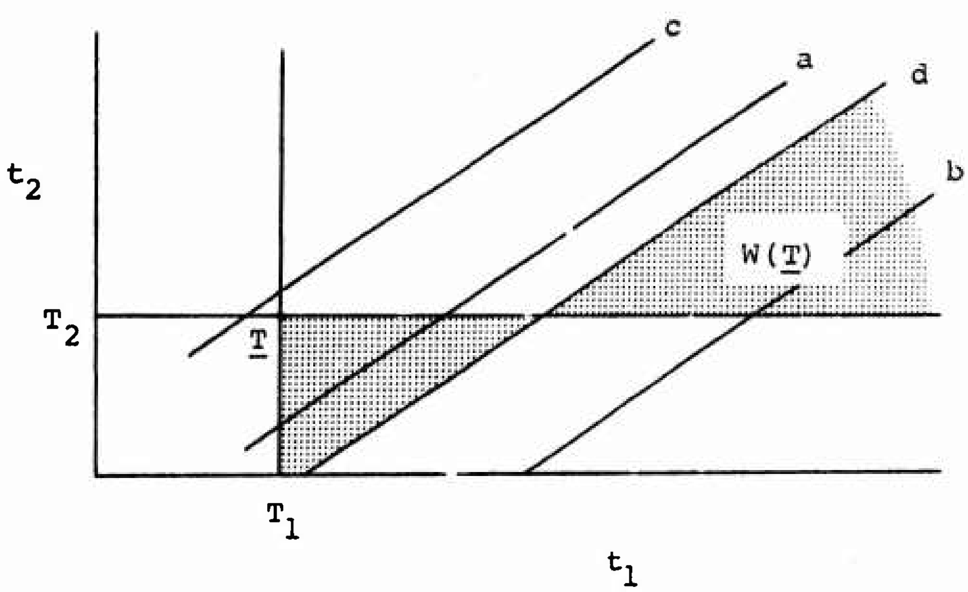

Figure D.1d SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 2'

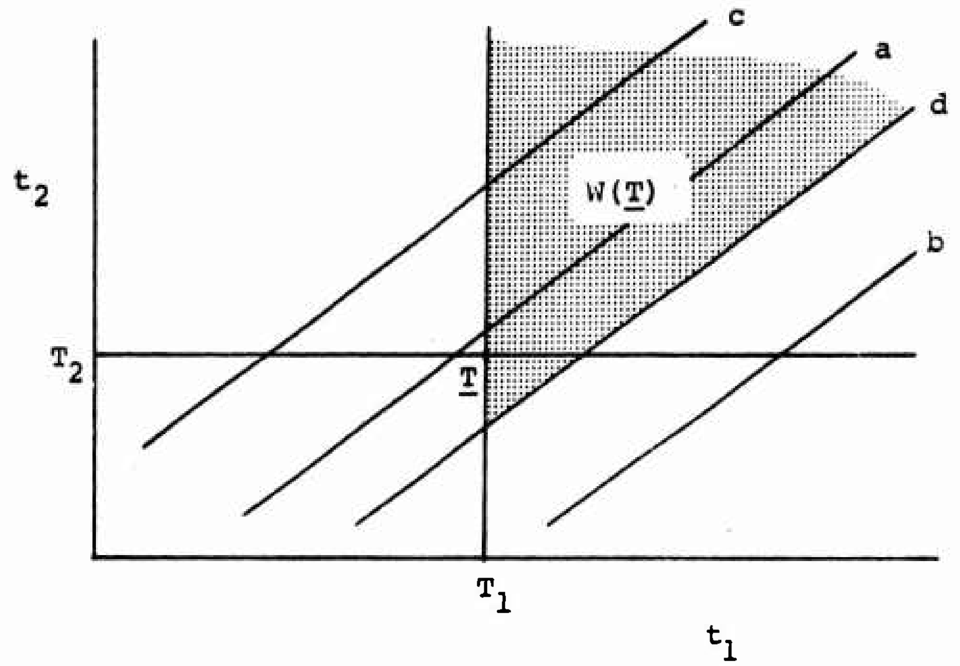

Figure D.1e SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 3

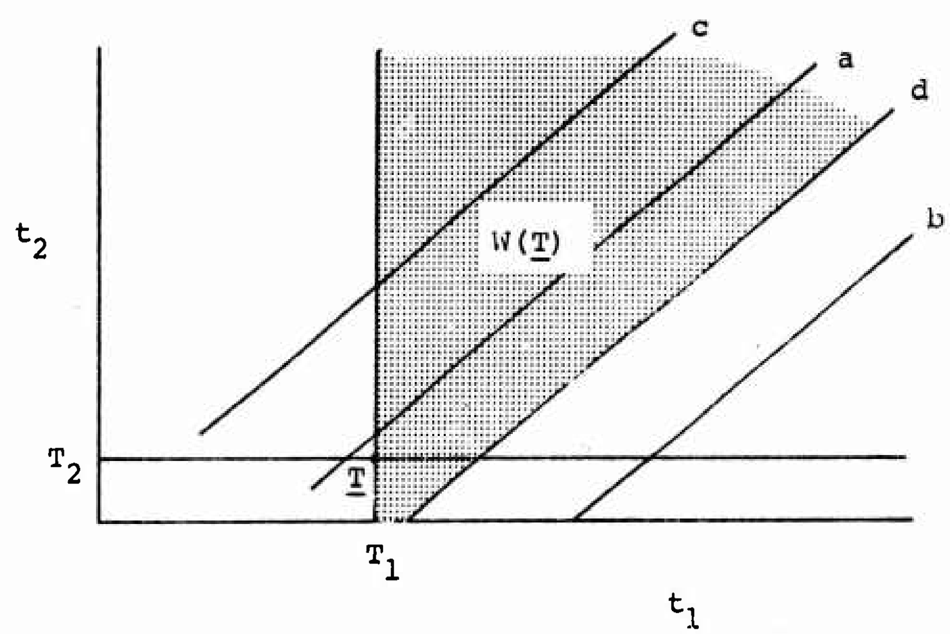

Figure D.1f SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 3'

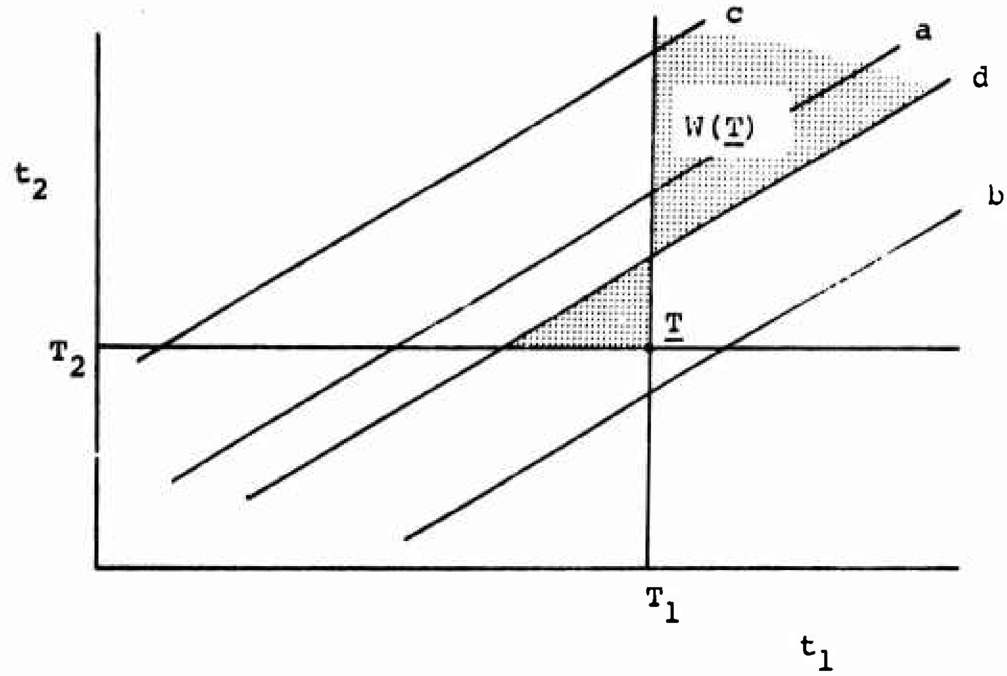

Figure D.1g SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 4

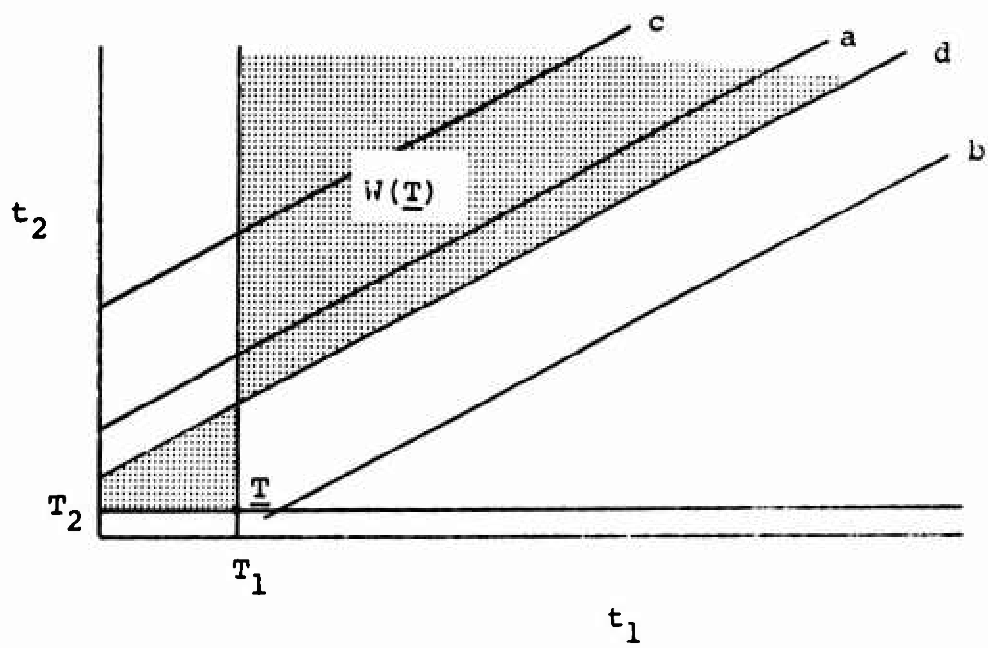

Figure D.1h SR-CSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 4'

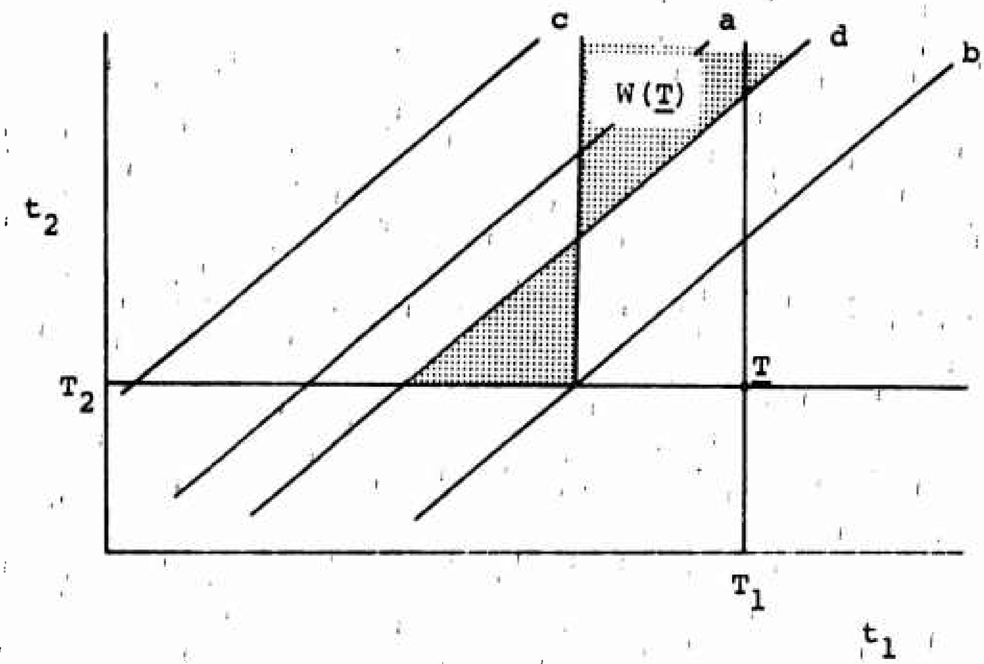

Figure D.1i SR-CSI Model W(T) for $\beta_1 < \beta_2$ Case 5

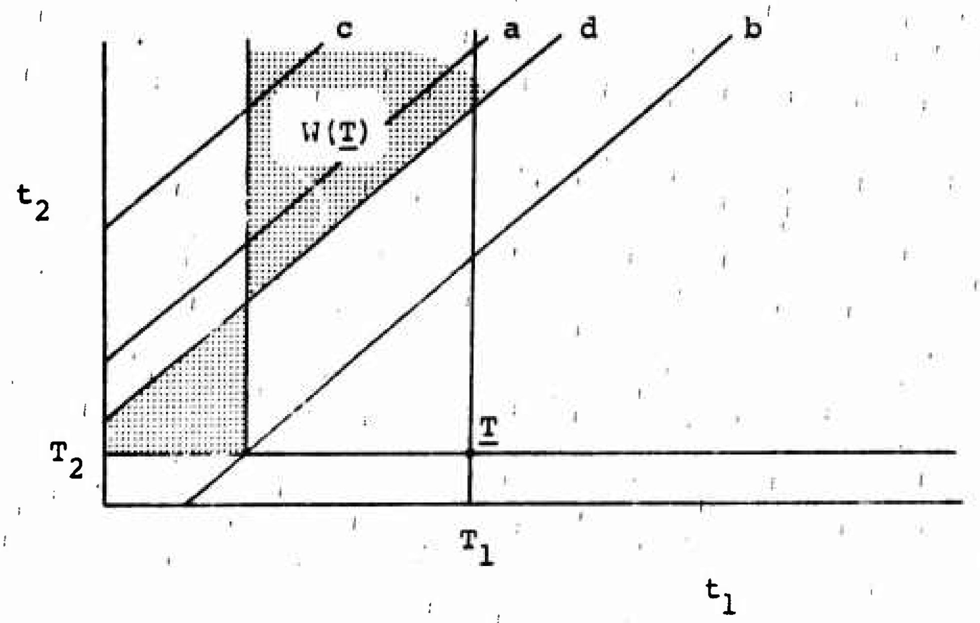

Figure D.1j SR-CSI Model W(T) for $\beta_1 < \beta_2$ Case 5'

optimal search must be unbounded as the available searching time grows without bound.

For any search allocation $\underline{T}$, let

T_{1b} = the abscissa of line b at $t_2 = T_2$,

T_{2c} = the ordinate of line c at $t_1 = T_1$.

T_{2d} = the ordinate of line d at $t_1 = T_1$,

T_{1d} = the abscissa of line d at $t_2 = T_2$,

T_d^1 = t_1 intercept of line d,

T_d^2 = t_2 intercept of line d.

Also, let $HQ(T_1,T_2)$ be the $H(\underline{T})$ function for case Q. Using (D.9) and the above implicit definitions of $W(\underline{T})$ direct integration yields

$$H1(T_1,T_2) = H2(T_1,T_{2c}) \ , \tag{D.11a}$$

$$H2(T_1,T_2) = \frac{1-\beta_2}{1-\beta_1\beta_2}\left\{p_1 e^{-k_1T_1-\beta_2k_2T_{2d}} - \beta_1 p_2 e^{-\beta_1k_1T_1-k_2T_{2d}}\right\} \tag{D.11b}$$

$$- \left\{p_1 e^{-k_1T_1-\beta_2k_2T_2} - p_2 e^{-\beta_1k_1T_1-k_2T_2}\right\} ,$$

$$H3(T_1,T_2) = \frac{1-\beta_2}{1-\beta_1\beta_2}\left\{p_1 e^{-k_1T_1-\beta_2k_2T_{2d}} - \beta_1 p_2 e^{-\beta_1k_1T_1-k_2T_{2d}}\right\} \tag{D.11c}$$

$$H4(T_1,T_2) = \frac{1-\beta_1}{1-\beta_1\beta_2}\left\{p_2e^{-k_2T_2-\beta_1k_1T_{1d}}-\beta_2p_1e^{-\beta_2k_2T_2-k_1T_{1d}}\right\}$$

(D.11d)

$$-\left\{p_2e^{-k_2T_2-\beta_1k_1T_1}-p_1e^{-\beta_2k_2T_2-k_1T_1}\right\} \quad ,$$

$$H5(T_1,T_2) = H4(T_{1b},T_2) \quad , \tag{D.11e}$$

and

$$H1'(T_1,T_2) = H2'(T_1,T_{2c}) \quad , \tag{D.12a}$$

$$H2'(T_1,T_2) = \frac{1-\beta_1}{1-\beta_1\beta_2}\left\{-\beta_2p_1e^{-k_1T_d^1} + p_2e^{-\beta_1k_1T_d^1}\right\}$$

$$+ p_1e^{-k_1T_1}\left(1-e^{-\beta_2k_2T_2}\right) \tag{D.12b}$$

$$- p_2e^{-\beta_1k_1T_1}\left(1-e^{-k_2T_2}\right) \quad ,$$

$$H3'(T_1,T_2) = \frac{1-\beta_1}{1-\beta_1\beta_2}\left\{p_2e^{-\beta_1k_1T_d^1}-\beta_2p_1e^{-k_1T_d^1}\right\}$$

(D.12c)

$$+ p_1e^{-k_1T_1} - p_2e^{-\beta_1k_1T_1} \quad ,$$

$$H4'(T_1,T_2) = \frac{1-\beta_2}{1-\beta_1\beta_2}\left\{p_1 e^{-\beta_2 k_2 T_d^2} - \beta_1 p_2 e^{-k_2 T_d^2}\right\}$$

$$+ p_2 e^{-k_2 T_2}\left(1-e^{-\beta_1 k_1 T_1}\right) \qquad \text{(D.12d)}$$

$$- p_1 e^{-\beta_2 k_2 T_2}\left(1-e^{-k_1 T_1}\right) ,$$

$$H5'(T_1,T_2) = H4'(T_{1b},T_2). \qquad \text{(D.12e)}$$

Consider an allocation, $\underline{T}^o$, in the case 1 or 1' region. If the corresponding $T_{2c} > 0$, there exist feasible allocations, $\underline{T}^+ = (T_1 + \Delta , T_{2c})$ for small positive Δ, in the case 2 or 2' region for which $W(\underline{T}^+)$ are proper subsets of $W(\underline{T}^o)$. Therefore, such $\underline{T}^o$ allocations cannot be optimal. Consequently, case 1 or 1' region allocations can only be optimal if no allocation with both T_1 and T_2 positive in the case 2 or 2' regions is feasible. By symmetry, case 5 or 5' region allocations can only be optimal if no case 4 or 4' region allocation with both T_1 and T_2 positive is feasible. Also, since H3 and H3' are decreasing functions of T_1 independent of T_2, case 3 or 3' region allocations can be optimal only if $T_2 = 0$. Therefore, for large T the optimal allocation must be in the case 2 or case 4 region.

Let us denote the partial derivatives of H2 with respect to T_1 and T_2 by $H2_1$ and $H2_2$. Then,

$$H2_1 = -k_1\left\{p_1 e^{-k_1T_1}\left(e^{-\beta_2k_2T_{2d}}-e^{-\beta_2k_2T_2}\right)\right. \tag{D.13a}$$

$$\left. -\beta_1p_2e^{-\beta_1k_1T_1}\left(e^{-k_2T_{2d}}-e^{-k_2T_2}\right)\right\} ,$$

and

$$H2_2 = k_2\left\{\beta_2p_1e^{-k_1T_1-\beta_2k_2T_2}-p_2e^{-\beta_1k_1T_1-k_2T_2}\right\} . \tag{D.13b}$$

It can easily be shown that both $H2_1$ and $H2_2$ are strictly negative for (T_1,T_2) in the case 2 region. Therefore, if we restrict (T_1,T_2) to the case 2 region and compute the allocation of T which minimizes H2, the resulting conditionally optimal H2*(T) function will be strictly decreasing in T. So any optimal allocation in the case 2 region must lie along the boundary $T_1 + T_2 = T$.

Conditionally optimal points in the case 2 region are characterized by

$$T_1 + T_2 = T, \text{ and} \tag{D.14a}$$

$$H2_1 = H2_2. \tag{D.14b}$$

Using (D.13) and eliminating T_1 from (D.14) we obtain

$$(1-\beta_2)e^{(1-\beta_1\beta_2)rk_1x}-(1-\frac{\beta_2k_2}{k_1})e^{(1-\beta_2)(k_1-\beta_2k_2)rx} \tag{D.15}$$

$$-\beta_2(\frac{k_2}{\beta_1k_1}-1)e^{-(1-\beta_2)(k_2-\beta_1k_1)rx} = 0 ,$$

where

$$x = T_2 - T_{2d},$$

$$r = \frac{k_2}{(1-\beta_1)k_1 + (1-\beta_2)k_2} .$$

Along, $T_1 + T_2 = T$ in the case 2 region, $H2_1$ can be shown to be strictly increasing while $H2_2$ is strictly decreasing. Further, these two functions cross exactly once in this interval. Therefore, (D.15) has exactly one root in the interval $(0, \frac{-\ln\beta_1}{(1-\beta_2)k_2})$ which provides the unique conditional solution in the case 2 region.

Note that (D.15) does not involve p_1, p_2, or T. Therefore, the position of the conditional solution in the case 2 region relative to line d is independent of the prior probabilities and the amount of available searching time. So the case 2 region conditional solution lies along a straight line parallel to lines a, b, c and d.

The conditional solution in the case 4 region is identical to that above for the case 2 region except that the roles of the boxes are interchanged.

In the case 3 or case 3' region $\frac{\partial H}{\partial T_1} < 0$ while $\frac{\partial H}{\partial T_2} = 0$. Thus the conditionally optimal solution for (T_1,T_2) between lines a and d is at or as near as possible to line d.

The conditional solution for the case 2' region is obtainable from marginal analysis similar to that for the

2 region. Corresponding to (D.15) we obtain

$$
\begin{aligned}
& e^{-k_1(w-T_2)} - \beta_1\beta_2 e^{-\beta_1 k_1(w-T_2)} \\
& -\left(1 - \frac{\beta_2 k_2}{k_1}\right) e^{-k_1 w + (k_1 - \beta_2 k_2)T_2} \\
& -\beta_2\left(\frac{k_2}{k_1} - \beta_1\right) e^{-\beta_1 k_1 w - (k_2 - \beta_1 k_1)T_2} = 0 \quad ,
\end{aligned}
\tag{D.16}
$$

where

$$w = T - T_c.$$

(B.16) depends on T but not on the prior probabilities. Therefore, the trace of conditional solutions in the case 2' region is not a straight line. But the shape of this trace is independent of the prior probabilities. The case 4' region conditional solution can be obtained by reversing the roles of the boxes in the case 2' solution.

For small values of T the solution of the marginal conditional $\frac{\partial H}{\partial T_1} = \frac{\partial H}{\partial T_2}$ along $T_1 + T_2 = T$ (from (D.15) or (D.16)) may violate the nonnegativity constraints T_1, $T_2 \geq 0$. In this case $\frac{dH}{dT_1}$ is monotonic in the feasible portion of the case 2 or case 2' region (or case 4 or case 4' region). Therefore, the conditional solution for the case 2 or case 2' region (or case 4 or case 4' region) is the point where $T_1 + T_2 = T$ intersects the boundary of

the feasible region nearest to the point where $\frac{\partial H}{\partial T_1} = \frac{\partial H}{\partial T_2}$.

Consider the variation of the integrand of (D.9), $y(t_1, t_2)$, along any line $s(\underline{t}) = e$. If

$$t_1' = t_1 + (1-\beta_2)rz,$$

and

$$t_2' = t_2 + (1-\beta_1)\frac{k_1}{k_2} rz,$$

$s(t_1',t_2') = s(t_1,t_2)$ so that $\underline{t}'$ represents points along $s(\underline{t}) = e$ parameterized by z. Also $t_1' + t_2' = t_1 + t_2 + z$. Direct substitution yields

$$y(\underline{t}') = y(\underline{t})e^{-\frac{k_1k_2(1-\beta_1\beta_2)}{(1-\beta_1)k_1+(1-\beta_2)k_2}z} . \qquad (D.17)$$

Therefore, the conditional optimal values of H decrease exponentially for the case 2 and case 4 regions. Hence, the optimal trajectory cannot switch from one conditional trajectory to the other in the region with T sufficiently large that the conditional trajectories lie in the case 2 and case 4 regions. Switching between conditional trajectories can, however, occur for smaller values of T.

The Algorithm for the SR-CSI Model

The following algorithm for computing the conditional trajectory corresponding to the case 2 or case 2' region is based on the above development. The case 4 or case 4' region conditional trajectory can be computed by this

algorithm by interchanging the roles of the boxes.

1. Compute the distance, x, which the case 2 region conditional trajectory lies above line d for large T from (D.15). (x is known to lie in the interval $(0, \frac{-\ln\beta_1}{(1-\beta_2)k_2})$.)

2. Compute the initial amounts of time, T_i^o, which must be allocated to each box before the other box receives a positive allocation.

   a. If $0 \leq c$, $T_1^o = \dfrac{\ln \dfrac{\beta_2 p_1}{p_2}}{(1-\beta_1)k_1}$, $T_2^o = 0$.

   b. If $c < 0$, $T_1^o = 0$ and T_2^o is determined by the condition $\frac{\partial H}{\partial T_1} = \frac{\partial H}{\partial T_2}$ along the T_2 axis.

      (1) If $0 \leq d$, $T_2^o = x$.

      (2) If $d < 0$, T_2^o is the root of

$$k_1\left\{p_1\left(1-e^{-\beta_2 k_2 T_2}\right)-\beta_1 p_2\left(1-e^{-k_2 T_2}\right)\right\} + k_2\left\{\beta_2 p_1 e^{-\beta_2 k_2 T_2}-p_2 e^{-k_2 T_2}\right\} = 0 \tag{D.18}$$

      in the interval (0,x).

3. If $T \leq T_1^o + T_2^o$, the conditional solution is

   $T_i^* = \min\{T, T_i^o\}$, for $i = 1,2$.

4. If $T > T_1^o + T_2^o$, both T_1^* and T_2^* are positive. The conditional solution is determined by $\frac{\partial H}{\partial T_1} = \frac{\partial H}{\partial T_2}$ along $T_1 + T_2 = T$.

a. If $T \geq x + \frac{d}{(1-\beta_1)k_1}$,

$T_2^* = T_{2d} + x$ and

$T_1^* = T - T_2^*$,

where x is the solution to (D.15) computed in step 1.

b. If $T < x + \frac{d}{(1-\beta_1)k_1}$, T_2^* is the root of (D.16), and $T_1^* = T - T_2^*$.

(This root is known to lie in the interval $(0,x)$.)

THE SR-ACSI PROBLEM

The adaptive complete search information version search problem can be analyzed by the same approach as used for the SR-CSI problem. Since the results for the CSI and ACSI versions are very similar, only the differences in formulas and results will be included here. ACSI version contacts occur in both boxes unless $t_1 + t_2 > T$. Therefore, the ACSI objective function corresponding to (D.1) is

$$Z = \iint_{t_1+t_2 \leq T} \max_{i=1,2} \{p_i(\underline{t},\underline{T})\}\, d\overline{F}(\underline{t}). \qquad \text{(D.19)}$$

The "wrong guess" region for the ACSI version, $W(\underline{T})$, are the portions of the corresponding CSI version $W(\underline{T})$ which

are in the $t_1 + t_2 > T$ half-plane. Figure D.2 shows the 10 possible $W(\underline{T})$ region shapes for the ACSI version corresponding to the cases shown in Figure D.1 for the CSI version.

Let

X_{ib} = T_i value at the intersection of line b and $T_1 + T_2 = T$,

X_{ic} = T_i value at the intersection of line c and $T_1 + T_2 = T$,

X_{id} = T_i value at the intersection of line d and $T_1 + T_2 = T$.

Then, corresponding to (D.11) for the CSI version we obtain for the ACSI version

$$H1(T_1,T_2) = H2(X_{1c},X_{2c}) \tag{D.20a}$$

$$\begin{aligned} H2(T_1,T_2) = &\left[\frac{1-\beta_2}{1-\beta_1\beta_2} - \frac{k_1}{k_1-\beta_2k_2}\right] p_1 e^{-k_1X_{1c}-\beta_2k_2X_{2c}} \\ &- \left[\frac{1-\beta_2}{1-\beta_1\beta_2} - \frac{k_1}{\beta_1k_1-k_2}\right] \beta_1 p_2 e^{-\beta_1k_1X_{1c}-k_2X_{2c}} \\ &+ \frac{\beta_1k_1p_1}{k_1-\beta_2k_2} e^{-k_1T_1-\beta_2k_2T_2} \\ &- \frac{k_2p_2}{\beta_1k_1-k_2} e^{-\beta_1k_1T_1-k_2T_2} \quad , \end{aligned} \tag{D.20b}$$

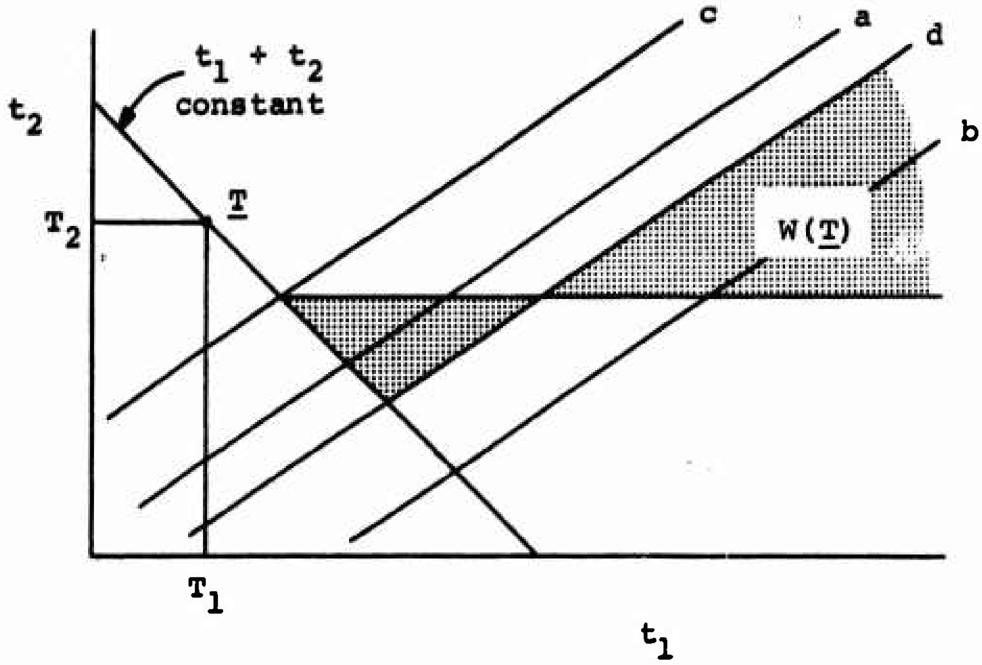

Figure D.2a SR-ACSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 1

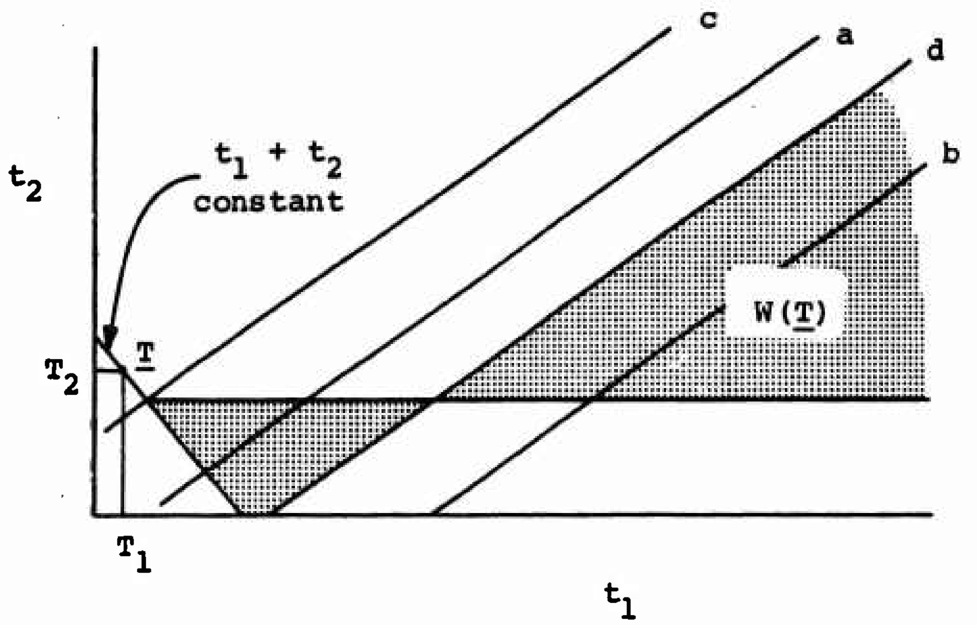

Figure D.2b SR-ACSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 1'

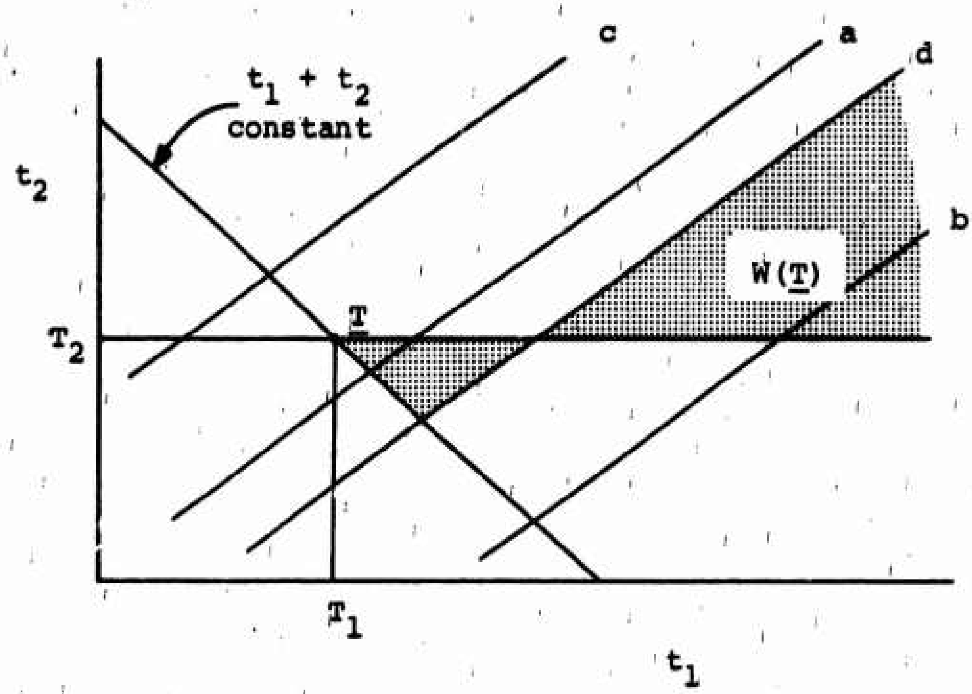

Figure D.2c SR-ACSI Model W(T) for $\beta_1 < \beta_2$ Case 2

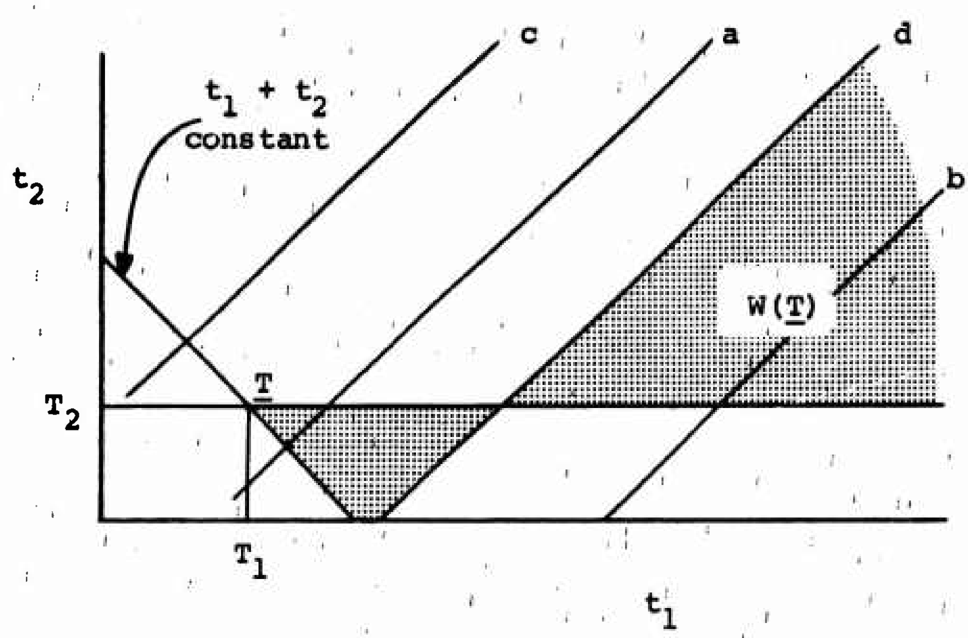

Figure D.2d SR-ACSI Model W(T) for $\beta_1 < \beta_2$ Case 2'

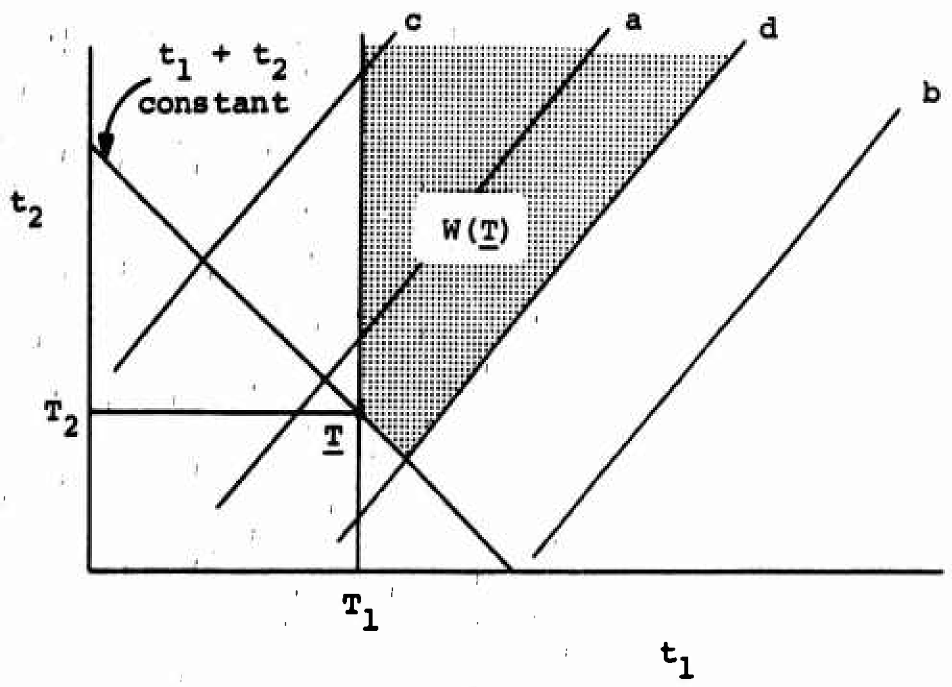

Figure D.2e SR-ACSI Model W(T) for $\beta_1 < \beta_2$ Case 3

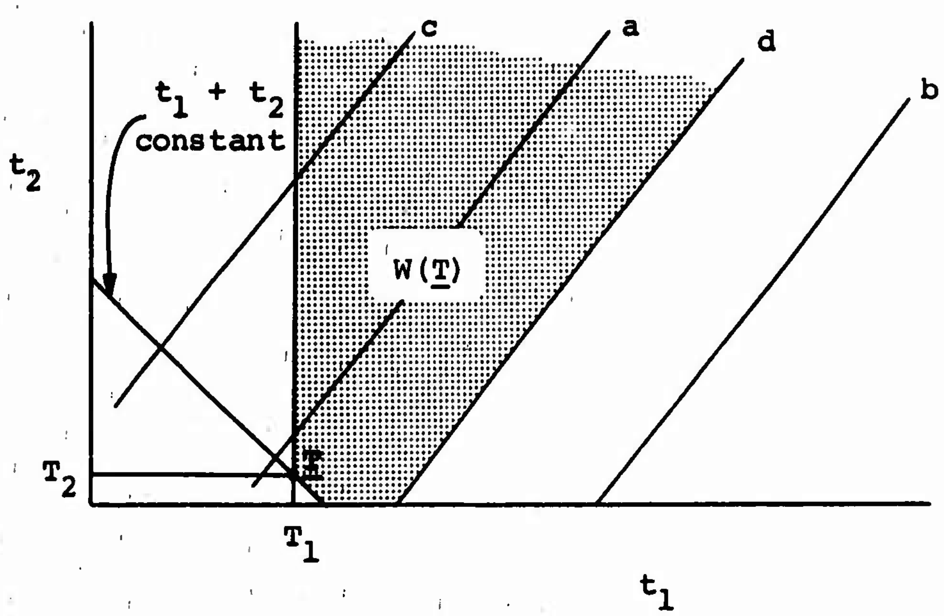

Figure D.2f SR-ACSI Model W(T) for $\beta_1 < \beta_2$ Case 3'

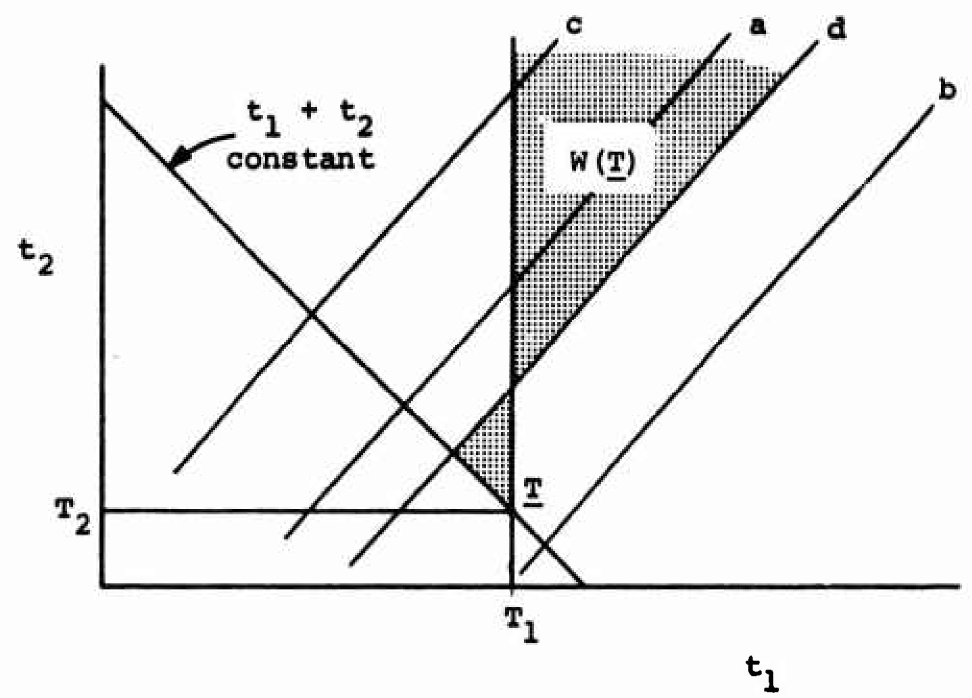

Figure D.2g SR-ACSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 4

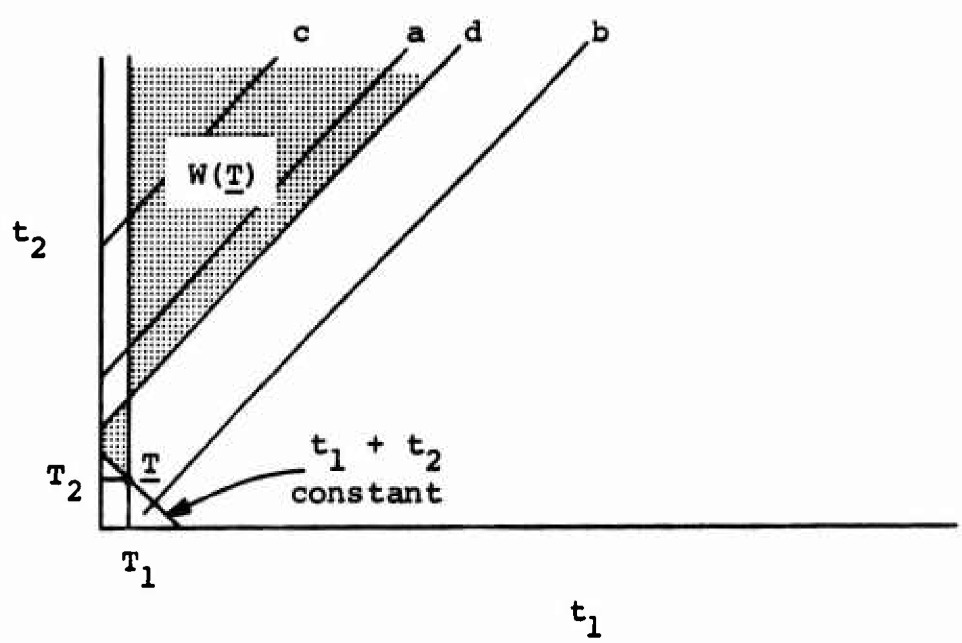

Figure D.2h SR-ACSI Model W($\underline{T}$) for $\beta_1 < \beta_2$ Case 4'

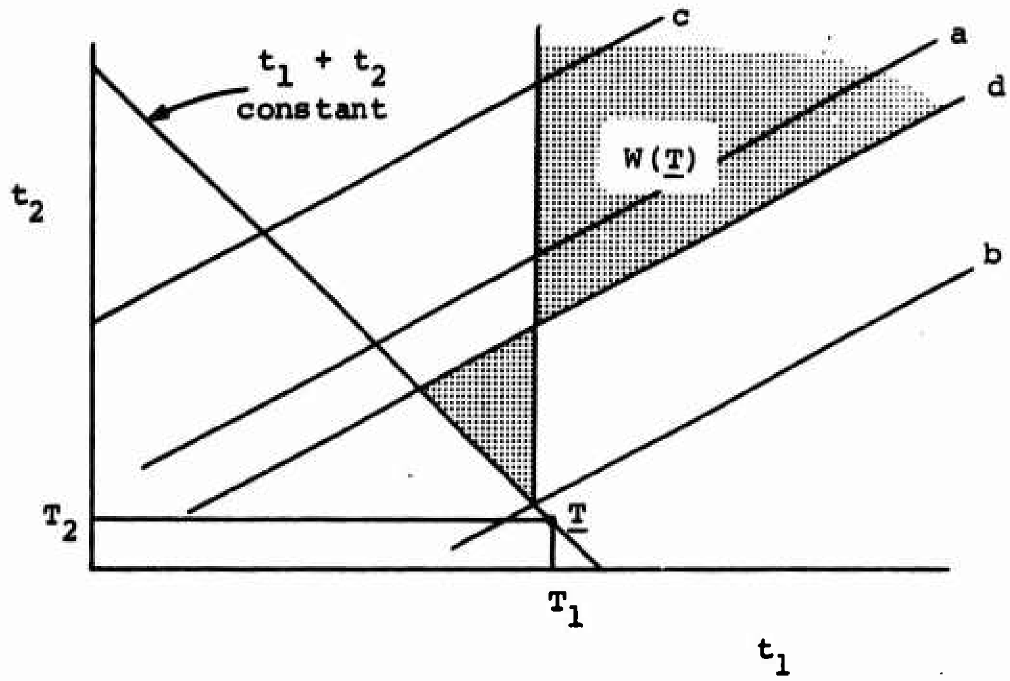

Figure D.2i SR-ACSI Model W(T̲) for $\beta_1 < \beta_2$ Case 5

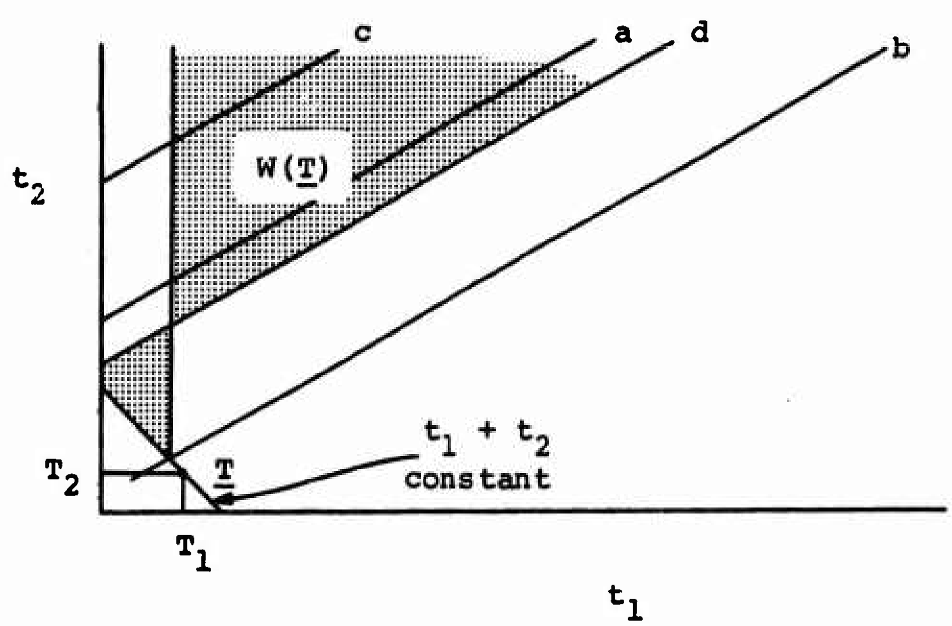

Figure D.2j SR-ACSI Model W(T̲) for $\beta_1 < \beta_2$ Case 5'

$$H3(T_1,T_2) = \left[\frac{1-\beta_2}{1-\beta_1\beta_2} - \frac{k_1}{k_1-\beta_2k_2}\right]p_1e^{-k_1X_{1d}-\beta_2k_2X_{2d}}$$

$$-\left[\frac{1-\beta_2}{1-\beta_1\beta_2} - \frac{k_1}{\beta_1k_1-k_2}\right]\beta_1p_2e^{-\beta_1k_1X_{1d}-k_2X_{2d}}$$

$$+\frac{k_1p_1}{k_1-\beta_2k_2}e^{-k_1T_1-\beta_2k_2T_2}$$

$$-\frac{k_1\beta_1p_2}{\beta_1k_1-k_2}e^{-\beta_1k_1T_1-k_2T_2} \quad , \tag{D.20c}$$

$$H4(T_1,T_2) = \left[\frac{1-\beta_1}{1-\beta_1\beta_2} - \frac{k_2}{k_2-\beta_1k_1}\right]p_2e^{-k_2X_{2b}-\beta_1k_1X_{1b}}$$

$$-\left[\frac{1-\beta_1}{1-\beta_1\beta_2} - \frac{k_2}{\beta_2k_2-k_1}\right]\beta_2p_1e^{-\beta_2k_2X_{2b}-k_1X_{1b}}$$

$$+\frac{\beta_2k_2p_1}{k_2-\beta_1k_1}e^{-k_2T_2-\beta_1k_1T_1}$$

$$-\frac{k_1p_1}{\beta_2k_2-k_1}e^{-\beta_2k_2T_2-k_1T_1} \quad , \tag{D.20d}$$

$$H5(T_1,T_2) = H4(X_{1b},X_{2b}) \quad , \tag{D.20e}$$

$$H1'(T_1,T_2) = H2'(X_{1c},X_{2c}) \quad , \tag{D.20f}$$

$$H2'(T_1,T_2) = \frac{(1-\beta_1)^2}{1-\beta_1\beta_2}\, p_1 \left[\frac{\beta_1 p_2}{\beta_2 p_1}\right]^{\frac{\beta_1}{1-\beta_1}}$$

$$+ \frac{\beta_2 k_2 p_1}{k_1-\beta_2 k_2} \left\{ e^{-k_1T_1-\beta_2k_2T_2} - e^{-k_1T} \right\} \qquad \text{(D.20g)}$$

$$- \frac{k_2 p_2}{\beta_1 k_1 - k_2} \left\{ e^{-\beta_1 k_1 T} - e^{-\beta_1 k_1 T_1 - k_2 T_2} \right\} ,$$

$$H3'(T_1,T_2) = \frac{1-\beta_1}{1-\beta_1\beta_2} \left[p_2 e^{-\beta_1 T_d} - \beta_2 p_1 e^{-T_d} \right]$$

$$+ \frac{\beta_2 k_2 p_1}{\beta_2 k_2 - k_1}\, e^{-T} + \frac{k_2 p_2}{\beta_1 k_1 - k_2}\, e^{-\beta_1 T} \qquad \text{(D.20h)}$$

$$+ \frac{k_1 p_1}{k_1 - \beta_2 k_2}\, e^{-k_1T_1-\beta_2k_2T_2}$$

$$+ \frac{k_1 \beta_1 p_2}{k_2 - \beta_1 k_1}\, e^{-\beta_1 k_1 T_1 - k_2 T_2} ,$$

$$H4'(T_1,T_2) = \frac{(1-\beta_2)^2}{1-\beta_1\beta_2} p_2 \left[\frac{\beta_2 p_1}{\beta_1 p_2}\right]^{\frac{\beta_2}{1-\beta_2}}$$

$$+ \frac{\beta_1 k_1 p_2}{k_2-\beta_1 k_1}\left\{e^{-k_2T_2-\beta_1k_1T_1} - e^{-k_2T}\right\} \qquad \text{(D.20i)}$$

$$- \frac{k_1 p_1}{\beta_2 k_2 - k_1}\left\{e^{-\beta_1 k_1 T} - e^{-\beta_1k_1T_1-k_2T_2}\right\} ,$$

$$H5'(T_1,T_2) = H4'(X_{1b},X_{2b}) , \qquad \text{(D.20j)}$$

where

$$T_d = \frac{\ln \frac{\beta_2 p_1}{\beta_1 p_2}}{(1-\beta_1)k_1} .$$

Note that for the case 1, case 1', case 5 and case 5' regions H is independent of the allocation, (T_1,T_2), along the constraint line $T_1 + T_2 = T$. That is, all feasible allocations with $T_1 + T_2 = T$ in one of these regions yield the same value of H. Within the case 2 or case 2' region $\frac{\partial H}{\partial T_1} = 0$ while $\frac{\partial H}{\partial T_2} < 0$. Therefore, the conditional solution confined to the case 2 or case 2' region is at or as near as possible to the boundary line c. Similarly, the case 4 or case 4' conditional solution is at or as near as possible to line b. And finally, in the case 3 or case

3' region $\frac{\partial H}{\partial T_1} < 0$ while $\frac{\partial H}{\partial T_2} = 0$. Therefore, the conditionally optimal solution in the case 3 region is at or as near as possible to line d.

These results may be summarized as follows:

1. An allocation, (T_1,T_2), between lines b and c can be optimal only if

   i. T is less than one of the nonnegative intercepts of lines b and c

   and

   ii. (T_1,T_2) is on one of the axes.

   That is, there exist two conditional trajectories consisting of the positive quadrant portions of lines b and c and the segments of the axes connecting these half-lines to the origin. An optimal solution can be found by considering the intersection of these conditional trajectories and the line $T_1 + T_2 = T$.

2. For $T_1 + T_2 = T$ the objective function is totally insensitive to the allocation in positive quadrant portion of the region above line c and in the positive quadrant portion of the region below line b.

The Algorithm for the SR-ACSI Model

Based on the above, the following algorithm computes the case 2 or case 2' region conditional trajectory for the ACSI version of the model. The corresponding case 4 or case 4' region conditional trajectory can be computed

by interchanging the roles of the boxes.

1. Determine the coordinates of the intersection of line c with the positive coordinate axes.

a. If $p_2 \leq \beta_2 p_1$, $T_1^o = \dfrac{\ln \dfrac{\beta_2 p_1}{p_2}}{(1-\beta_1)k_1}$, $T_2^o = 0$.

b. If $\beta_2 p_1 < p_2$, $T_1^o = 0$, $T_2^o = \dfrac{\ln \dfrac{p_2}{\beta_2 p_1}}{(1-\beta_2)k_2}$.

2. The conditional trajectory is

a. For $T \leq T_1^o + T_2^o$

$T_i^* = \min\{T, T_i^o\}$ for $i = 1,2$.

b. For $T_1^o + T_2^o < T$,

$$T_1^* = \frac{(1-\beta_2)k_2T + \ln \dfrac{\beta_2 p_1}{p_2}}{(1-\beta_1)k_1 + (1-\beta_2)k_2} ,$$

$$T_2^* = \frac{(1-\beta_1)k_1T - \ln \dfrac{\beta_2 p_1}{p_2}}{(1-\beta_1)k_1 + (1-\beta_2)k_2} .$$

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